18.01 Single Variable Calculus
Fall 2006

SCOTT'S NOTES ON GRADING METHOD: I'VE TRIED TO BE FAIR IN ASSESSING INCORRECT ANSWERS, DUE TO A SIMPLE MISTAKE (~~[illegible]~~ WHICH I HAVE LABELED AS ALGEBRA ERROR, DEDUCTING 2 PTS FOR EACH WHICH STILL HAD THE SAME PROCEDURE TO ARRIVE AT A SOLUTION OTHER TRIVIAL MISTAKES (I.E. NOT CORE TO THE MATERIAL IN THE COURSE WERE SIMILARLY WEIGHTED)

CALCULUS 18.01

$\frac{181}{250}$ = FINAL SCORE = $\boxed{72.4\%}$

EXAM TIME, 3 HOURS

Q	Score	Value
1	13	15
2	6	10
3	8	10
4	10	10
5	15	15
6	13	15
7	11	15
8	11	15
9	8	10
10	13	15
11	4	15
12	16	20
13	4	10
14	11	15
15	10	15
16	6	10
17	10	10
18	12	15
19	6	10
TOTAL	181	250

18.01 Practice Final Exam

There are 19 problems, totaling 250 points. No books, notes, or calculators. This practice exam should take 3 hours.

Generally useful trigonometry:

$$\sin^2 x = \frac{1-\cos 2x}{2}; \qquad \cos^2 x = \frac{1+\cos 2x}{2}; \qquad \int \sec x = \ln(\sec x + \tan x)$$

$$\sec x = \frac{1}{\cos x}; \qquad \sin^2 x + \cos^2 x = 1; \qquad \tan^2 x + 1 = \sec^2 x$$

In a 30-60-90 right triangle, with hypotenuse 2, the legs are 1 and $\sqrt{3}$.

$$\sqrt{2} = 1.41 \qquad \sqrt{3} = 1.73 \qquad \pi = 3.14 \qquad \ln 2 = .69 \qquad \ln 10 = 2.3$$

~~Problem 1.~~ (15) Evaluate each of the following:

a) $\dfrac{d}{dx}\dfrac{\ln x}{x^2}$; simplify your answer.

b) $\dfrac{d}{du}\sqrt{3\sin^2 u + 2}$

c) $\dfrac{d^n}{dx^n}e^{kx}\Big|_{x=0}$, k constant.

~~Problem 2.~~ (10) Find the equation of the line tangent to the graph of $x^2y^2 + y^3 = 2$ at the point $(1,1)$ on the graph. (Give the equation in the form $y = mx + b$.)

S4 → **~~Problem 3.~~** (10) Using implicit differentiation, derive the formula for $D\cos^{-1} x$ by using the formula for $D\cos x$. (Let $y = \cos^{-1} x$.)

S4 → **~~Problem 4.~~** (10) Let $f(x) = \begin{cases} x^2 + x + a, & x \le 0 \\ bx + 2, & x > 0 \end{cases}$, a and b constants. Find all values of a and b for which $f(x)$ is differentiable.

~~Problem 5.~~ (15) On a night when the full moon is directly overhead, an outdoor Christmas tree 50 feet high is falling over. Its top is falling at the rate of 2 feet/sec, at the moment when it is 30 feet from the ground. At that moment, how rapidly is the shadow of the tree cast by the moon lengthening?

~~Problem 6.~~ (15) Find the area of the largest rectangle whose base lies along the x-axis and whose top corners lie on the parabola $y = 1 - x^2$.

~~Problem 7.~~ (15: 4,7,4) The graph of $y = y(x)$ has this property: at each point (x, y) on the graph, the normal line at that point passes through the fixed point $(1, 0)$. (The normal is the line perpendicular to the tangent line.)

a) Show that $y = y(x)$ satisfies the differential equation $y' = \dfrac{1-x}{y}$.

b) Using separation of variables, find all solutions to the differential equation. You can leave the solutions in implicit form, i.e., as equations connecting x and y.

c) Describe the curves which are their graphs. (You may have to use algebraic processes first (like completing the square) in order to change the equations into a form where you know what their graphs look like.)

~~Problem 8.~~ (15) The cup of a wine-glass has the shape formed by rotating the parabola $y = x^2$ about the y-axis; its upper rim is a circle of radius 1. How much wine does it hold?

~~Problem 9.~~ (10) Using the trapezoidal rule with three subdivisions ($n = 3$), estimate $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$. Do the work systematically, making a table of values first.

~~Problem 10.~~ (15: 7,8) Let $F(x) = \displaystyle\int_0^x e^{-t^2}\,dt$.

a) Find $F'(1)$ and $F''(1)$.

b) Express $\displaystyle\int_1^2 e^{-u^2/4}\,du$ in terms of values of $F(x)$.

~~Problem 11.~~ (15: 7,8) Between the two towers of a suspension bridge, each of the two main cables has the shape of the parabola $y = \frac{1}{10}x^2$ (units are kilometers). The two towers are 2 km. apart; the vertical cables from the main cable to the horizontal roadway are closely and equally spaced.

a) Set up a definite integral which gives the length of each main cable between the two towers.

b) What is the average length (to the nearest meter) of the vertical cables?

~~Problem 12.~~ (20: 10,10) Evaluate

a) $\displaystyle\int_0^1 \frac{dx}{x^2+3x+2}$; (begin by factoring the denominator). b) $\displaystyle\int x^2 \ln x\,dx$

~~Problem 13.~~ (10) Evaluate $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$ by making the substitution $x = \tan u$; remember the limits.

~~Problem 14.~~ (15) Starting at the point where $r = 1$, the point P moves counterclockwise along the polar curve $r = e^{\theta/2\pi}$, in such a way that the line segment OP makes one complete revolution. (Here O denotes the origin.)

Sketch the curve, and find the total area swept out by OP as it makes the revolution.

~~Problem 15.~~ (15) Evaluate (showing work):

a) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}$ b) $\displaystyle\lim_{x\to 1} \frac{(\ln x)^2}{x-1}$ c) $\displaystyle\lim_{x\to\infty} x^2 e^{-x}$

~~Problem 16.~~ (10) Evaluate $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}}$

~~Problem 17.~~ (10) For what values of p does $\displaystyle\sum_1^\infty \frac{n}{\sqrt{4+n^p}}$ converge? (Indicate reasoning.)

Problem 18. (15: 10,5)

a) Find by differentiating the function $f(x) = \sqrt{1+x}$ the first four non-zero terms of its Taylor series around $x = 0$. (Show work.)

b) Use the correct answer to (a) (or your own answer, if you don't know the correct answer) to calculate $\sqrt{1.2}$ to four decimal places.

Problem 19. (10) Find the Taylor series for $\tan^{-1} x$ around $x = 0$ by using term-by-term differentiation or integration on the appropriate geometric series. Give enough terms to make the pattern clear.

PROBLEM 2

CHECK!

find tan line at $(1,1)$ of $x^2y^2+y^3=2$

$$2xy^2+2x^2yy'+3y^2y'=2 \quad \times$$

SHOULD BE ZERO!

$$2-2xy^2=2x^2yy'+3y^2y'$$

$\boxed{-4}$

$$y'(2x^2y+3y^2)=2-2xy^2$$

$$y'=\frac{2-2xy^2}{2x^2y+3y^2}$$

$$m=\frac{2-2(1)(1)^2}{2(1)^2(1)+3(1)^2}=\frac{2-2}{2+3}=0$$

$\boxed{y=1}$

$y=mx+b$

$y=1$

PROBLEM 1

A) $$\frac{d}{dx}\frac{\ln x}{x^2}=\frac{u'v-uv'}{v^2}$$

$$\frac{\frac{1}{x}\cdot x^2-\ln x\cdot 2x}{x^4}=\boxed{\frac{x-2x\ln x}{x^4}}$$ ✓

B) $$\frac{d}{du}\sqrt{3\sin^2 u+2}=-\frac{1}{2\sqrt{3\sin^2 u+2}}\cdot(6\sin u)\cdot\cos u$$

$$=\boxed{-\frac{3\sin u\cos u}{\sqrt{3\sin^2 u+2}}}$$

-2 ALGEBRA ERROR

C) $$\left.\frac{d^n}{dx^n}e^{kx}\right|_{x=0}=k^ne^0=\boxed{k^n}$$ ✓

PROBLEM 3

$D\cos^{-1}x$

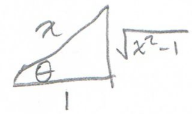
$y=\cos^{-1}x$

$-\csc(\cos^{-1}x)$

$\cos y = x$

$-\sin y\, y' = 1$

$y' = -\frac{1}{\sin y} = -\csc y$



ALGEBRA ERROR

$\boxed{-2}$

$= -\frac{1}{\sin(\cos^{-1}x)}$

$= \cancel{-\frac{1}{\frac{\sqrt{x^2-1}}{x}} = \boxed{-\frac{x}{\sqrt{x^2-1}}}}$

$= \boxed{-\frac{1}{\sqrt{x^2-1}}}$

PROBLEM 4

$f(x)$ is differentiable where $f_1(0)=g_1(0)$ & $f_1'(0)=g_1'(0)$

$f_1(0)=x^2+x+a$

$g_1(0)=6x+2$

$x^2+x+a=6x+2$

$a=2$

$f_1'(0)=2x+1$

$g_1'(0)=6$

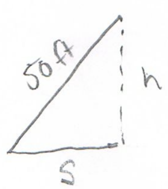
$2x+1=6$

$b=1$

@ $x=0$

The function is differentiable at $a=2$, $b=1$ ✓

PROBLEM 5



$\frac{dh}{dt} = 2\text{ft/sec}$

$50^2 = s^2 + h^2$

$\frac{d}{dt}\, s^2 = \frac{d}{dt} 50^2 - h^2$

$2s\frac{ds}{dt} = -2h\frac{dh}{dt}$

$\frac{ds}{dt} = \frac{-2h}{2s}\left(\frac{dh}{dt}\right)$

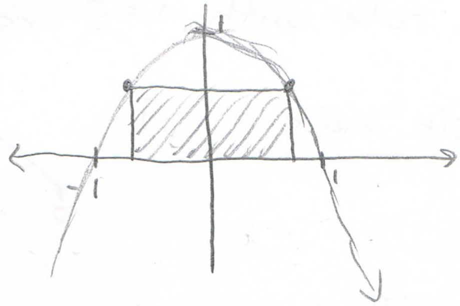
$\frac{ds}{dt} = \frac{-2(30)}{2(40)}(-2)$ ✓

$\frac{ds}{dt} = \frac{3}{2}$ ft/second

The shadow is lengthening by 1.5 ft/second

PROBLEM 6

Find the largest rectangle within: $y = 1 - x^2$



$A = h \cdot b$

$b = 2x$

$h = 1 - x^2$

$A = (2x)(1-x^2) = 2x - 2x^3$

$\frac{dA}{dx} = -6x^2 + 2$

$0 = -6x^2 + 2$

$\frac{1}{3} = x^2$

$x = \pm \frac{1}{\sqrt{3}}$ ✓

$1 - \left(\frac{1}{\sqrt{3}}\right)^2$

$1 - \frac{1}{3}$

The rectangle defined by $\left(\frac{1}{\sqrt{3}}, 2/3\right)$ as the top-right pt. is the largest

−2 FORGOT TO EXPRESS AS FINAL ANSWER

PROBLEM 8

$y = x^2$ What is the volume?

$dy = 2x\,dx$

$$\int_0^1 \underbrace{2\pi x^2}_{\text{Area of a disk}} \cdot \underbrace{2x\,dx}_{\text{thickness of disk}} = V$$ ✓

INCORRECT FORMULA FOR AREA −4

$$V = \int_0^1 4\pi x^3\,dx$$

$$= 4\pi \left.\frac{x^4}{4}\right|_0^1$$

$$= \pi \cdot 1 - \pi(0)$$

$$\boxed{= \pi}$$

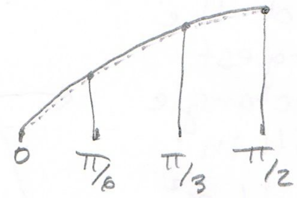
The volume is π units

PROBLEM 9

CHECK!

Using trap. rule with $n=3$ estimate

$$\int_0^{\pi/2} \sin^2 x \, dx$$



x	$\sin^2 x$
0	0
$\pi/6$	$1/\sqrt{2}$
$\pi/3$	$3/\sqrt{2}$
$\pi/2$	1

$$\frac{y_0+y_1}{2}\Delta x + \frac{y_1+y_2}{2}\Delta x + \frac{y_2+y_3}{2}\Delta x =$$

$$\left.\begin{array}{l} \sin\left(\frac{\pi}{3}\right) = \frac{3}{\sqrt{2}} \\ \sin\left(\pi/6\right) = \frac{1}{\sqrt{2}} \end{array}\right\} \times$$

WRONG VALUES FOR $\sin(\frac{\pi}{6})$ $\sin(\frac{\pi}{3})$

CORRECT TRAP RULE

$$\frac{0+\frac{1}{\sqrt{2}}}{2}\frac{\pi}{6} + \frac{2}{\sqrt{2}}\cdot\frac{\pi}{6} + \frac{1+\frac{3}{\sqrt{2}}}{2}\cdot\frac{\pi}{6}$$

$$= \frac{\pi}{12\sqrt{2}} + \frac{\pi}{3\sqrt{2}} + \frac{\pi(3+\sqrt{2})}{12\sqrt{2}}$$

$$= \frac{5\pi}{12\sqrt{2}} + \frac{\pi}{4} + \frac{\pi\sqrt{2}}{12\sqrt{2}} = \boxed{\frac{6\pi}{12\sqrt{2}} + \frac{\pi}{4}}$$

$\boxed{-2}$

PROBLEM 10

a) Let $F(x) = \int_0^x e^{-t^2} dt$

Find $F'(1)$ and $F''(1)$

$$F'(x) = e^{-x^2} \quad \therefore F'(1) = e^{-1} = \boxed{\frac{1}{e}}$$

$$F''(x) = -2xe^{-x^2} \quad \therefore F''(1) = \boxed{\frac{-2}{e}}$$

✓

b) Express $\int_1^2 e^{-u^2/4} du$ in terms of $F(x)$

CHECK!

Let $x = \frac{1}{2}u$

$dx = \frac{1}{2}du$

$$\int_?^? 2e^{-x^2} dx \qquad 2\int_{1/2}^{1} e^{-x^2} dx$$

$$\boxed{\frac{F(1)-F(1/2)}{2}}$$

DIVIDED INSTEAD OF MULTIPLIED

$\boxed{-2}$

PROBLEM 12

a) $\displaystyle\int_0^1 \frac{dx}{x^2+3x+2} =$

$$\int_0^1 \frac{dx}{(x+1)(x+2)} = \int_0^1 \frac{A}{x+1} + \int_0^1 \frac{B}{x+2}$$

$$A = \frac{1}{-1+2} = 1 \quad ✓$$
$$B = \frac{1}{-2+1} = -1$$

$$= \int_0^1 \frac{1}{x+1}\,dx - \int_0^1 \frac{1}{x+2} =$$

$$\ln|x+1|\Big|_0^1 - \ln|x+2|\Big|_0^1 =$$

$$\boxed{= \ln(2) - \ln(3)}$$

incorrect evaluation algebra error $\boxed{-2}$

b) $$\int \underbrace{x^2}_{dv} \underbrace{\ln x}_{u}\,dx = \frac{x^3 \ln x}{3} - \int \frac{x^{\cancel{3}\,2}\,dx}{3\,\cancel{x}}$$

$u = \ln x \qquad v = \frac{x^3}{3}$

$du = \frac{dx}{x} \qquad dv = x^2\,dx$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\,\frac{x^3}{3}$$

$$= \boxed{\frac{x^3 \ln x}{3} - \frac{x^3}{9}}$$

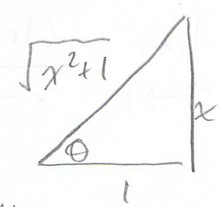
MISSI[NG] C: -2

PROBLEM 13

$$\int_0^1 \frac{dx}{(x^2+1)^2}$$

$x = \tan u$

$dx = \sec^2 u\,du$



$$\int_0^1 \frac{\sec^2 u\,du}{(\tan^2 u+1)^2} = \int_0^1 \frac{\sec^2 u\,du}{\sec u} = \int_0^1 \frac{1}{\cos u}\,du$$

algebra error, dropped square $\boxed{-2}$

MISSED WORK DUE TO ERROR $= \boxed{-4}$

$$= \ln|\sec u + \tan u|\Big|_0^1$$

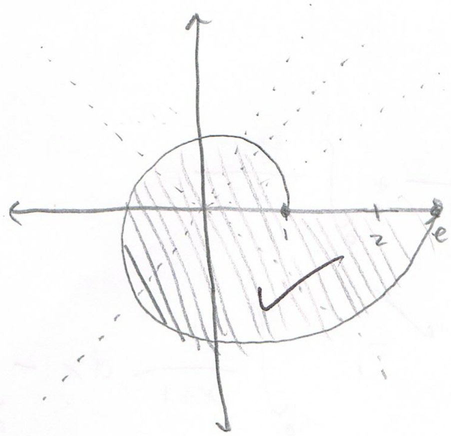
$$= \ln|\sec(\tan^{-1}x) + \tan(\tan^{-1}x)|\Big|_0^1$$

$$\ln\left|\sqrt{x^2+1} + x\right|\Big|_0^1$$

$$\boxed{= \ln(1+\sqrt{2})}$$

PROBLEM 14

$r = e^{\theta/2\pi}$



$\theta = \pi/2 \quad e^{\pi/2/2\pi} = e^{1/4}$

Find Area in curve

$dA = \frac{1}{2} r^2 \, d\theta$

$= \int_0^{2\pi} \frac{1}{2}(e^{\theta/2\pi}) d\theta$ ← forgot to square $\boxed{-2}$

$= \frac{1}{2} \int_0^{2\pi} e^{\theta/2\pi} d\theta$

$= \frac{2\pi}{2} e^{\theta/2\pi} \Big|_0^{2\pi} = \pi e^{\theta/2\pi} \Big|_0^{2\pi}$

algebra error $\boxed{-2}$

$= \pi e^1 - \pi e^0$

$\boxed{= \pi e}$

PROBLEM 15

A) $\lim_{x\to 0} \frac{\sin^2 x}{1-\cos^2 x} = \lim_{x\to 0} \frac{2\sin x \cos x}{1 + 2\cos x \sin x}$

should be $\sin^2 x$

$= \frac{0}{1} = \boxed{0}$ ✗

B) $\lim_{x\to 1} \frac{(\ln x)^2}{x-1} = \lim_{x\to 1} \frac{\frac{2\ln x}{x}}{1-1}$

$= \boxed{0}$ ✓

c) $\lim_{x\to\infty} x^2 e^{-x}$

$= \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$

$= \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = \boxed{0}$ ✓

PROBLEM 16) Evaluate

$$\int_1^\infty \frac{dx}{x^{3/2}} =$$

CONVERGES BECAUSE THE RADIUS OF CONVERGENCE FOR $\int_1^\infty \frac{dx}{x^n}$ $n \geq 1$

INCORRECT INTEGRATION $\boxed{-4}$

$$x\ \frac{-1}{2\sqrt{x}}\Big|_1^\infty$$

$$0-\left(-\frac{1}{2}\right) = \boxed{1/2}$$

PROBLEM 18)

Taylor's Formula $= \dfrac{f^n(0)}{n!}x^n$

$f(x) = \sqrt{1+x}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\quad f'(0) = 1/2$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\quad f''(0) = -1/4$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$ $\quad f'''(0) = \frac{3}{8}$

A) $\boxed{T(x) \doteq \frac{1}{0!}x^0 + \frac{1/2}{1!}x^1 - \frac{1/4}{2!}x^2 + \frac{3/8}{3!}}$

B) $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{3x^3}{42}$

$\doteq 1 + \frac{.2}{2} - \frac{(.2)^2}{8} + \frac{3(.2)^3}{42}$

$\doteq 1 + 0.1 - .005 + .00015$

$\doteq \boxed{1.0948}$

incorrect arithmatic $\boxed{-3}$

$.2 \cdot .2 \Rightarrow \frac{.04}{8} = .005$

$.2 \cdot .04 \Rightarrow \frac{.008}{6 \cdot 8} = \frac{.001}{6}$

$\doteq .00015$

P19 Find Taylor Series for $\tan^{-1} x$

$$\tan^{-1}(x) \doteq \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n$$

$$\boxed{= x + \frac{2x^3}{3!} + \frac{4x^5}{5!} + \frac{8x^7}{7!} + \frac{16x^9}{9!} \cdots}$$

X

INCORRECT APPROACH

$y = \tan^{-1}(x)$

$\tan y = x$

$\sec^2 y\, y' = 1$

$y' = \cos^2 y$

$y' = \cos^2 \tan^{-1} x$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 0$

$f'''(0) = 2$

$f''''(0) = 0$

$f^{(5)}(0) = 4$

$f(x) = \tan^{-1}(x)$

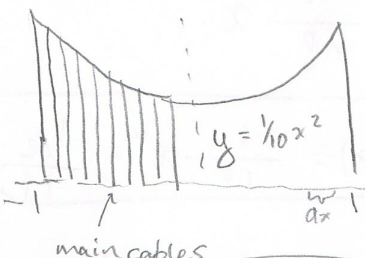
$f'(x) = \cos^2 \tan^{-1} x$

$f''(x) = -2\sin \tan^{-1} x \ldots$

$f'''(x) = -2\cos^2 \tan^{-1} x - 2s\ldots$

$f''''(x) = 4\sin \cos y - 4$

PROBLEM 11



A) $$\boxed{L(x) = \int_0^x \frac{x}{5}\, dx}$$

length of cable at x ($x=0$ is middle of bridge, units in km)

X

MISINTERPRETED QUESTION

(B) Average Length of cable =

$$\int_{-1}^{1} \frac{1}{10} x^2 \cdot d\cancel{x}$$

length of cable

each cable

total of all dx from $-1 \to 1$

$$\frac{\left.\frac{x^3}{\cancel{20}}\right|_{-1}^{1}}{2} = \frac{1}{40} + \frac{1}{40} = \frac{1}{2}$$

should be 30

incorrect integral

$\boxed{-4}$

$$\boxed{\text{Average length} = \frac{1}{40}\text{ km}}$$

PROBLEM 7

A) $y(x)$ all normal lines intersect with $(1,0)$

Show $y' = \frac{1-x}{y}$ satisfies this

slope of tan line

equation of tan line at (x_0, y_0)

$y_0 - y_0 = y'(x_0 - x_0)$

~~$y = y'(x-$~~

eq'n of normal lin

slope of normal line $-\frac{1}{y'} = \frac{y}{x-1} =$

$y = \frac{1}{y'}(1-x)$

$$\boxed{y' = \frac{1-x}{y}}$$

✓

B)

$y'y = 1-x$

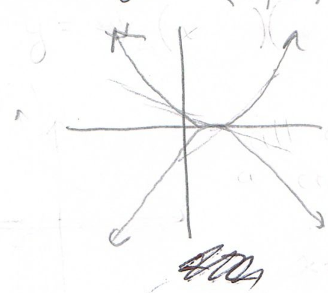
$\frac{dy}{dx}y = \int 1-x$

$\int y\,dy = \int (1-x)\,dx$

$$\boxed{\frac{y^2}{2} = x - \frac{x^2}{2} + C}$$

✓

C) The graphs are hyperbola with a turning point at $(1,0)$



✗

PROBLEM 17

For what values of p does

$$\sum_{1}^{\infty} \frac{n}{\sqrt{4+n^p}}$$

converge?

$$\int_{1}^{\infty} \frac{x\,dx}{\sqrt{4+x^p}}$$

if $\frac{p}{2} - 1 >$ the limit wi converge

$$\boxed{p > 4}$$

✓

$$\doteq \int_{1}^{\infty} \frac{x}{\sqrt{4 * x^p}}\,dx \Rightarrow \lim_{x\to\infty} \frac{x}{\sqrt{4+x^p}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^p}} = \lim_{x\to\infty} = C$$

(1) If the summation converges/diverges the integral of this form will as well

this will converge if the limit of the top reaches ∞ slower than the bottom

if this is zero, the sum will converge, otherwise it will not

the 4 can be dropped since it tends to zero whe $x \to \infty$