SOLUTIONS TO THE 18.02 FINAL EXAM

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1) For each of (a)-(e) below: If the statement is true, write TRUE. If the statement is false, write FALSE. (Please do not use the abbreviations T and F.) No explanations are required in this problem.

(a) (5 pts.) If $f(x, y)$ is a continuously differentiable function on $\mathbb{R}^2$, and $\dfrac{\partial^2 f}{\partial x^2} = 0$ at every point of $\mathbb{R}^2$, then there exist constants a and b such that $f(x, y) = ax + b$ for all x and y.

Solution: FALSE. The values of a and b could depend on y. The function $f(x, y) = y$ is a counterexample.

(b) (5 pts.) The annulus R in $\mathbb{R}^2$ defined by $9 \le x^2 + y^2 \le 16$ is simply connected.

Solution: FALSE. The circle of radius 3.5 centered at the origin gives a closed curve in R that cannot be shrunk to a point within R.

(c) (5 pts.) If $f(x, y)$ is a function whose second derivatives exist and are continuous everywhere on $\mathbb{R}^2$, and

$$f(0, 0) = f_x(0, 0) = f_y(0, 0) = f_{xx}(0, 0) = f_{yy}(0, 0) = 0$$

and $f_{xy}(0, 0) \ne 0$, then f has a saddle point at $(0, 0)$.

Solution: TRUE. This follows from the second derivative test. In the notation of that test, we are given $A = 0$, $B \ne 0$, and $C = 0$, so $AC - B^2 < 0$, so f has a saddle point at $(0, 0)$.

(d) (5 pts.) If A is a 3×3 matrix, and $\mathbf{b}$ is a column vector in $\mathbb{R}^3$, and the square system $A\mathbf{x} = \mathbf{b}$ has more than one solution $\mathbf{x}$, then $\det A = 0$.

Solution: TRUE. If $\det A \ne 0$, then there would have been one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

(e) (5 pts.) If $\mathbf{F}$ is a continuously differentiable 3D vector field on $\mathbb{R}^3$ such that $\operatorname{curl} \mathbf{F} = \mathbf{0}$ everywhere, and A and B are two points in $\mathbb{R}^3$, then the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same for every piecewise smooth path C from A to B.

Solution: TRUE. The region $\mathbb{R}^3$ is simply connected, so all six conditions for conservativeness are equivalent, and these are two of them.

2) (a) (5 pts.) For which pairs of real numbers (a, b) does the matrix

$$A = \begin{pmatrix} a & b & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

have an inverse?

Solution: We have

$$\det A = 0 + b + 0 - 0 - 0 - 0 = b,$$

so A is invertible if and only if $b \neq 0$. (There is no condition on a.)

(b) (10 pts.) For such pairs (a, b), compute A^{-1}.
(Its entries may depend on a and b. *Suggestion:* Once you have the answer, check it!)

Solution: Compute nine 2×2 determinants to get the matrix of minors:

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -b \\ b & a & 0 \end{pmatrix}$$

Apply the checkerboard signs to get the matrix of cofactors:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & b \\ b & -a & 0 \end{pmatrix}$$

Take the transpose to get the adjoint matrix:

$$\begin{pmatrix} 0 & 0 & b \\ 1 & 0 & -a \\ 0 & b & 0 \end{pmatrix}$$

Divide by $\det A$ to get the inverse matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1/b & 0 & -a/b \\ 0 & 1 & 0 \end{pmatrix}.$$

3) Let L be the line that passes through $(7, 3, -1)$ and is perpendicular to the plane P with equation $2x + y - z = 6$. Find the point where L intersects P.

Solution: The vector $\mathbf{N} = \langle 2, 1, -1 \rangle$ is a normal vector to the plane, so it is in the direction of L. Thus L has a parametrization

$$\mathbf{r}(t) = \langle 7, 3, -1 \rangle + t\langle 2, 1, -1 \rangle = \langle 7 + 2t, 3 + t, -1 - t.$$

Substitute this into the equation of the plane to find the value of t for which $\mathbf{r}(t)$ lies on the plane:

$$\begin{aligned} 2(7 + 2t) + (3 + t) - (-1 - t) &= 6 \\ 6t + 18 &= 6 \\ t &= -2. \end{aligned}$$

Thus the intersection point is

$$\mathbf{r}(-2) = \langle 3, 1, 1 \rangle.$$

4) A particle is moving in the plane so that its distance from the origin is increasing at a constant rate of 2 meters per second, and its argument θ is increasing at a rate of 3 radians per second. At a time when the particle is at $(4, 3)$, what is its velocity vector?

Solution: Differentiating

$$x = r\cos\theta$$
$$y = r\sin\theta$$

yields

$$\frac{dx}{dt} = (\cos\theta)\frac{dr}{dt} + (-r\sin\theta)\frac{d\theta}{dt}$$
$$\frac{dy}{dt} = (\sin\theta)\frac{dr}{dt} + (r\cos\theta)\frac{d\theta}{dt}.$$

At the given time, $r = \sqrt{4^2 + 3^2} = 5$, $\cos\theta = 4/5$, and $\sin\theta = 3/5$. Substituting shows that at that time,

$$\frac{dx}{dt} = \frac{4}{5}\cdot 2 + (-5\cdot\frac{3}{5})3 = -\frac{37}{5}$$
$$\frac{dy}{dt} = \frac{3}{5}\cdot 2 + (5\cdot\frac{4}{5})3 = \frac{66}{5},$$

so the velocity vector is $\left\langle -\frac{37}{5}, \frac{66}{5} \right\rangle$, or equivalently $\langle -7.4, 13.2 \rangle$.

5) Let $f(x, y) = x^2 - xy + y^6$. Find the *minimum* value of the directional derivative $D_{\mathbf{u}}f$ at the point $(2, 1)$ as $\mathbf{u}$ varies over unit vectors in the plane.

Solution: We have $\nabla f = \langle 2x - y, -x + 6y^5 \rangle$, which at $(2, 1)$ equals $\langle 3, 4 \rangle$. Then $D_{\mathbf{u}}f = (\nabla f)\cdot\mathbf{u}$, which is minimized when $\mathbf{u}$ is in the direction *opposite* to $\langle 3, 4 \rangle$, i.e., when $\mathbf{u} = -\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$. For this $\mathbf{u}$, we obtain

$$D_{\mathbf{u}}f = |\nabla f||\mathbf{u}|\cos\pi = -|\nabla f| = -5.$$

6) Find an equation for the tangent plane to the surface $x^2 + y^2 + 3z = 8$ at the point $(2, 1, 1)$.

Solution: Let $f(x, y, z) = x^2 + y^2 + 3z$. The tangent plane is perpendicular to $\nabla f = \langle 2x, 2y, 3 \rangle$, which at $(2, 1, 1)$ equals $\langle 4, 2, 3 \rangle$. So $\langle 4, 2, 3 \rangle$ is a normal vector for the tangent plane, so the plane has equation

$$4x + 2y + 3z = c$$

for some number c. Substituting $(2, 1, 1)$ shows that $c = 13$, so the tangent plane is

$$4x + 2y + 3z = 13.$$

7) Let S be the part of the sphere $x^2 + y^2 + z^2 = 9$ where x, y, z are all positive. Find the minimum *value* of the function

$$\frac{8}{x} + \frac{8}{y} + \frac{1}{z}$$

on S, or explain why it does not exist.

Solution: Define

$$\begin{aligned} f(x, y, z) &:= \frac{8}{x} + \frac{8}{y} + \frac{1}{z} \\ g(x, y, z) &:= x^2 + y^2 + z^2 \\ c &:= 9. \end{aligned}$$

We use Lagrange multipliers to find the minimum value. First,

$$\begin{aligned} \nabla f &= \left\langle -\frac{8}{x^2}, -\frac{8}{y^2}, -\frac{1}{z^2} \right\rangle \\ \nabla g &= \langle 2x, 2y, 2z \rangle . \end{aligned}$$

So the Lagrange multiplier system is

$$\begin{aligned} x^2 + y^2 + z^2 &= 9 \\ -\frac{8}{x^2} &= \lambda(2x) \\ -\frac{8}{y^2} &= \lambda(2y) \\ -\frac{1}{z^2} &= \lambda(2z). \end{aligned}$$

Since x, y, z are nonzero, we can solve for λ in each equation to obtain

$$\lambda = -\frac{4}{x^3} = -\frac{4}{y^3} = -\frac{1}{2z^3}.$$

Take reciprocals and multiply by -4 to obtain

$$x^3 = y^3 = 8z^3.$$

Take cube roots to obtain $x = y = 2z$. Substituting into the first equation (the constraint equation) yields

$$\begin{aligned} (2z)^2 + (2z)^2 + z^2 &= 9 \\ 9z^2 &= 9 \\ z &= 1 \qquad \text{(since } z > 0\text{)} \\ x &= 2z = 2 \\ y &= 2z = 2. \end{aligned}$$

So this yields $(2, 2, 1)$.

We also check

- points on $g = c$ where $\nabla g = \mathbf{0}$. The only point with $\nabla g = \mathbf{0}$ is $(0, 0, 0)$, which is not on $g = c$.
- points on $g = c$ where f or g is not differentiable. There are no such points in S.

- behavior at ∞. Not applicable, since S is bounded.
- behavior at boundary. We have constraint inequalities $x > 0$, $y > 0$, $z > 0$. As the boundary is approached, one of x, y, z tends to 0 (from the right), so $f(x, y, z)$ tends to $+\infty$.

By the geometry, f must have a minimum. The only possibility is that it occurs at $(2, 2, 1)$. So the minimum value is $f(2, 2, 1) = 4 + 4 + 1 = 9$.

8) Suppose that $s(x, y)$ and $t(x, y)$ are differentiable functions such that it is possible to express x as a differentiable function of s and t. Express $\left(\frac{\partial x}{\partial s}\right)_t$ in terms of the partial derivatives s_x, s_y, t_x, t_y.

Solution 1 (differentials): We have

$$\begin{aligned} ds &= s_x\,dx + s_y\,dy \\ dt &= t_x\,dx + t_y\,dy \end{aligned}$$

but we need dx in terms of ds and dt. So we treat ds and dt as known, and dx and dy as unknown, and solve for dx. Namely, multiply the first equation by t_y and the second equation by s_y, and subtract to get

$$\begin{aligned} t_y\,ds - s_y\,dt &= (t_y s_x - s_y t_x)\,dx \\ dx &= \frac{t_y}{t_y s_x - s_y t_x}\,ds - \frac{s_y}{t_y s_x - s_y t_x}\,dt. \end{aligned}$$

Finally, $\left(\frac{\partial x}{\partial s}\right)_t$ is the coefficient of ds in this combination, so

$$\left(\frac{\partial x}{\partial s}\right)_t = \frac{t_y}{t_y s_x - s_y t_x}.$$

Solution 2 (two-Jacobian rule): By the two-Jacobian rule,

$$\begin{aligned} \left(\frac{\partial x}{\partial s}\right)_t &= \frac{\partial(x,t)/\partial(x,y)}{\partial(s,t)/\partial(x,y)} \\ &= \frac{\begin{vmatrix} x_x & x_y \\ t_x & t_y \end{vmatrix}}{\begin{vmatrix} s_x & s_y \\ t_x & t_y \end{vmatrix}} \\ &= \frac{\begin{vmatrix} 1 & 0 \\ t_x & t_y \end{vmatrix}}{\begin{vmatrix} s_x & s_y \\ t_x & t_y \end{vmatrix}} \\ &= \frac{t_y}{s_x t_y - t_x s_y}. \end{aligned}$$

9) Let R be the parallelogram in $\mathbb{R}^2$ bounded by the lines

$$y = x - 1, \qquad y = x - 3, \qquad y = 5 - 2x, \qquad y = 7 - 2x.$$

Evaluate $\displaystyle\iint_R \frac{2x+y}{x-y}\,dx\,dy$.

Solution: Use the change of variable $u = 2x + y$ and $v = x - y$. The boundary curves become the lines $v = 1$, $v = 3$, $u = 5$, $u = 7$, so the corresponding region is a rectangle in the uv-plane. We compute

$$\begin{aligned}
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\
&= -3 \\
\frac{\partial(x,y)}{\partial(u,v)} &= -\frac{1}{3} \\
dx\,dy &= \frac{1}{3}\,du\,dv.
\end{aligned}$$

Thus

$$\begin{aligned}
\iint_R \frac{2x+y}{x-y}\,dx\,dy &= \int_{v=1}^{3}\int_{u=5}^{7} \frac{u}{v}\,\frac{1}{3}\,du\,dv \\
&= \frac{1}{3}\int_{v=1}^{3} \left.\frac{u^2/2}{v}\right|_{u=5}^{7} dv \\
&= \frac{1}{3}\int_{v=1}^{3} \frac{12}{v}\,dv \\
&= 4\int_{v=1}^{3} \frac{1}{v}\,dv \\
&= 4\ln 3.
\end{aligned}$$

10) Let $\mathbf{F}$ be a vector field defined everywhere on $\mathbb{R}^3$ except the origin, pointing radially outward with magnitude $|\mathbf{F}| = 1/\rho$, where ρ is the distance to the origin. Let S_R be the sphere of radius R centered at the origin.

(a) (15 pts.) Compute the outward flux of $\mathbf{F}$ across S_R, in terms of the positive number R.

Solution: At each point of S_R, the vectors $\mathbf{F}$ and $\mathbf{n}$ are in the same direction, so $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}||\mathbf{n}| = |\mathbf{F}| = 1/\rho = 1/R$. So the flux is

$$\begin{aligned}\oiint_{S_R} \mathbf{F}\cdot\mathbf{n}\,dS &= \oiint_{S_R} \frac{1}{R}\,dS \\ &= \frac{1}{R}\oiint_{S_R} dS \\ &= \frac{1}{R}(4\pi R^2) \\ &= 4\pi R.\end{aligned}$$

(b) (10 pts.) Let T be the 3D region between the spheres S_1 and S_3, i.e., the region where

$$1 \le \sqrt{x^2+y^2+z^2} \le 3.$$

What is $\iiint_T \operatorname{div}\mathbf{F}\,dV$?

Solution: The boundary of T consists of the two spheres, but S_1 has the wrong orientation, so the (extended) divergence theorem yields

$$\begin{aligned}\iiint_T \operatorname{div}\mathbf{F}\,dV &= \oiint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS - \oiint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS \\ &= 12\pi - 4\pi \qquad \text{(by part (a))} \\ &= 8\pi.\end{aligned}$$

11) Let R be the solid triangle in $\mathbb{R}^2$ with vertices at $(0,1)$, $(2,3)$, and $(0,3)$. Let C be its boundary traversed counterclockwise. Let $\mathbf{F} = x^2(\mathbf{i}-\mathbf{j})$.

(a) (20 pts.) Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ by converting it to a double integral over the region R.

Solution: We have

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(x^2) \\ &= -2x - 0 \\ &= -2x.\end{aligned}$$

By Green's theorem,

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R \operatorname{curl}\mathbf{F}\,dA \\
&= \int_{x=0}^{2}\int_{y=x+1}^{3} -2x\,dy\,dx \\
&= \int_{x=0}^{2} -2x(3-(x+1))\,dx \\
&= \int_{x=0}^{2} -2x(2-x)\,dx \\
&= \int_{x=0}^{2} 2x(x-2)\,dx \\
&= \int_{x=0}^{2} 2x^2-4x\,dx \\
&= \frac{2}{3}x^3-2x^2\Big|_0^2 \\
&= \frac{2}{3}2^3-8 \\
&= -\frac{8}{3}.
\end{aligned}$$

(b) (5 pts.) Is there a function $f(x,y)$ whose gradient equals $\mathbf{F}$ everywhere? (Explain your reasoning.)

Solution 1: NO. If $\mathbf{F}$ were a gradient, then $\oint_C \mathbf{F}\cdot d\mathbf{r}$ would have been 0.

Solution 2: NO. If $\mathbf{F}$ were a gradient, then $\operatorname{curl}\mathbf{F}$ would have been 0 everywhere, but above we found that $\operatorname{curl}\mathbf{F} = -2x$.

12) Set up an iterated integral in *cylindrical* coordinates whose value is the moment of inertia of a solid spherical planet P of radius a and constant density δ with respect to an axis through the center of the planet. (Assume that the center of the planet is at the origin, and that the axis of rotation is the z-axis.)

Do not evaluate the integral!

Solution: The equation for the sphere is $x^2+y^2+z^2 = a^2$, which is equivalent to $r^2+z^2 = a^2$. So, given r, the range for z is from $-\sqrt{a^2-r^2}$ to $\sqrt{a^2-r^2}$. The moment of inertia is

$$\begin{aligned}
I &= \iiint\limits_P (\text{distance to axis})^2\, dm \\
&= \iiint\limits_P r^2\, \delta\, dV \\
&= \iiint\limits_P r^2\, \delta\, dz\, r\, dr\, d\theta \\
&= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \int_{z=-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^2\, \delta\, dz\, r\, dr\, d\theta \\
&= \delta \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \int_{z=-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3\, dz\, dr\, d\theta.
\end{aligned}$$

13) Let S be the lower hemisphere defined by $x^2+y^2+z^2 = 1$ and $z \le 0$. Let $\mathbf{F} = \langle y+xz, 5-x, 2e^x \rangle$. Compute the outward (i.e., downward) flux of $\operatorname{curl} \mathbf{F}$ across S.

Solution 1: The boundary of S is the circle C parametrized by $\mathbf{r}(t) := \langle \cos t, \sin t, 0 \rangle$ for $t \in [0, 2\pi]$, except that its orientation is wrong relative to the outward normal of S. So the flux is

$$\begin{aligned}
\iint\limits_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} &= -\oint_C \mathbf{F} \cdot d\mathbf{r} \qquad \text{(by Stokes' theorem)} \\
&= -\int_0^{2\pi} \langle \sin t, 5-\cos t, 2e^{\cos t} \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\, dt \\
&= -\int_0^{2\pi} (-\sin^2 t + 5\cos t - \cos^2 t)\, dt \\
&= -\int_0^{2\pi} (5\cos t - 1)\, dt \\
&= -\left. (5\sin t - t) \right|_0^{2\pi} \\
&= -(-2\pi) \\
&= 2\pi.
\end{aligned}$$

Solution 2: By Stokes' theorem, the flux integral of $\operatorname{curl} \mathbf{F}$ can be converted to a line integral of $\mathbf{F}$ on the unit circle in the plane $z = 0$, which by Stokes' theorem again can be converted to a flux integral of $\operatorname{curl} \mathbf{F}$ across the disk D given by $x^2+y^2 \le 1$ in the plane $z = 0$. The unit normal for D compatible with the orientation of the circle (clockwise when viewed from the top) is again the *downward* unit normal, i.e., $\mathbf{n} = -\mathbf{k}$, so we get

$$\begin{aligned}
\iint\limits_S (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} &= \iint\limits_D (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} \\
&= \iint\limits_D (\operatorname{curl}\mathbf{F}) \cdot (-\mathbf{k})\, dS \\
&= \iint\limits_D - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y + xz & 5 - x \end{vmatrix} dS \\
&= \iint\limits_D 2\, dS \\
&= 2\,\mathrm{Area}(D) \\
&= 2\pi.
\end{aligned}$$