18.03 - Differential Equations

Final Exam Written: Nov 29, 2011

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	Score	Total
Q1	10	10
Q2	7	10
Q3	4	10
Q4	9	10
Q5	9	10
Q6	7	10
Q7	6	10
Q8	5	10
Q9	1	10
Q10	2	10
TOTAL	60	100

= 60 %

SCOTT'S NOTES ON GRADING:

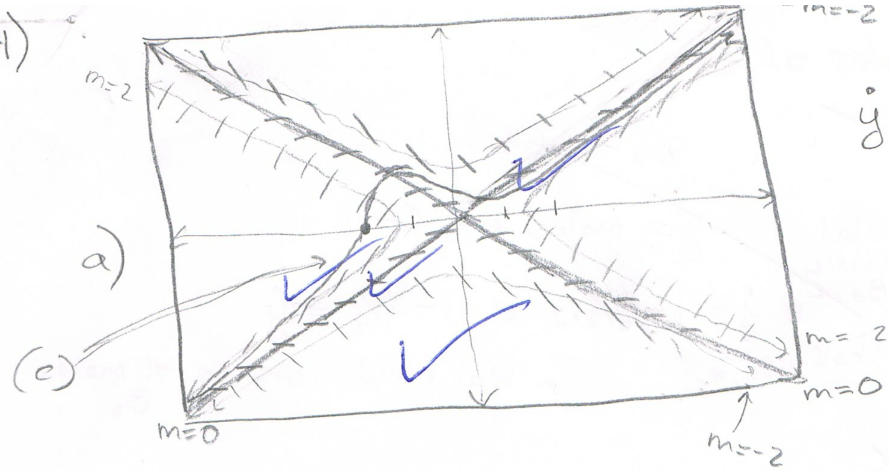
AS NO RUBRIC WAS PROVIDED FOR ALLOCATING MARKS, I CHOSE TO ALLOCATE THEM EVENLY. I OMITTED 2(b) FROM BOTH GRADING AND TOTAL BECAUSE IT HAD A TYPO WHICH COMPLETELY CHANGED THE PROVIDED SOLUTION. ALGEBRAIC ERRORS WERE COUNTED AS $\boxed{-1}$ PT, ALL OTHERS WERE GENERALLY THE WHOLE VALUE OF THAT PARTICULAR SUB-QUESTION.

2 ← represent grade for sub-questions (a, b, c, etc.)

$\boxed{8}$ ← represent grade for total (sum of all circled marks)

1)

a)



$\dot{y} = x^2 - y^2$

c) $\dot{y} = (-2)^2 - (0)^2 = 4$

d) $f(100) \doteq 100$

$\boxed{10}$

e) $f(a) = -a$ if max @ $x = a$

f)

$1.5 \cdot 1.5 = 2.25$
$.75$
1.50
$.625$

$h \doteq 0.5$

x	y	m	hm
-2	0	4	2
-1.5	2	-1.75	-0.8775
-1	1.125		

2)

a)

$\dot{x} = x(x^2 - 3x + 2)$

$\dot{x} = x(x-2)(x-1)$

2

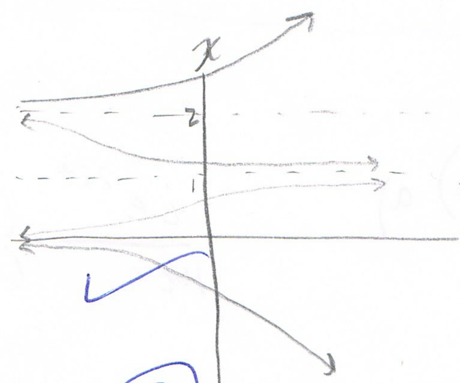
$(-1)(-3)(-2) = -$

$(0.5)(-1.5)(-0.5) = +$

$(1.5)(-0.5)(0.5) = -$

$2.5(0.5)(1.5) = +$

b)



2

c) $\ddot{x} = 0 = 2\dot{x} - \cancel{6x\dot{x}} + 3x^2\dot{x}$

$0 = \dot{x}(3x^2 \cancel{+6}x + 2)$

$= \dot{x}($ $)$

$\dfrac{+6 \pm \sqrt{36-24}}{6} = \boxed{1 \pm \sqrt{12}}$

possible inflection values of x.

ALGERBRA ERROR -1

2

d) Let y = # moles of Ct

OMIT DUE TO TYPO IN QUESTION

$\dot{y} = 1 + \frac{B_0}{4}$ ← initial quantity of B_0

1 mole/unit time of Ct added

half-life, *2

$B_0 \rightarrow Ct$

y = moles of Ct

$\dot{y} = 1 - \frac{y}{2\cdot 2} = 1 - \frac{y}{4} = \dot{y}$

initial condition is # moles of B_0

e) $x\dot{y} + 3y = x$

1

$\dot{y} + \frac{3}{x}y = x$

$e^{3\ln x} y = \int e^{3\ln x}$ ✓ $u = x$, $v' = e^{3\ln x}$, $u' = 1$, $v = \frac{e^{3\ln x}}{3x}$

X WRONG INTEGRATING FACTOR SOL'N −2

$e^{3\ln x} y = \frac{e^{3\ln x}}{3} - \int \frac{e^{3\ln x}}{3x}\,dx$

$e^{3\ln x} y = \frac{e^{3\ln x}}{3} - \frac{e^{3\ln x}}{3x} + c$

$y = \frac{1}{3}\left(1 - \frac{1}{x}\right) + ce$

$1 = \frac{1}{3} - \frac{1}{1} + ce^{3\ln(1)}$

$\frac{5/3}{e^{3\ln 1}}$

$y = \frac{1}{3}\left(1 - \frac{1}{x}\right) + \frac{5/3}{e^{3\ln 1}} e^{3\ln x}$

3) a) $\frac{ie^{2it}}{1+i} \cdot \frac{(1-i)}{(1-i)} = \frac{ie^{2it} + e^{2it}}{2} = \frac{i\cos 2t - \sin 2t + \cos 2t + i\sin}{2}$

$= -\frac{1}{2}\sin(2t) + \frac{1}{2}\cos(2t)$

2

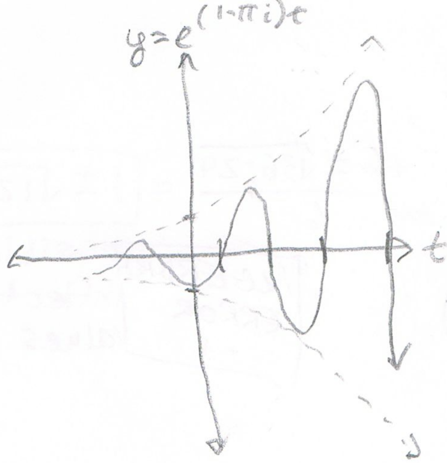
$\frac{1}{2}$, $-\frac{1}{2}$, ϕ

$A = \frac{1}{\sqrt{2}}$ ✓

$\omega = 2$ ✓

$\phi = -\pi/4$ or $5\pi/4$

X $\boxed{\pi/4}$

b) 

$-e^{t}\cos(\pi t)$ ✓

1

c) $\sqrt[3]{8i} =$

$|\sqrt[3]{8i}| = 2$ ✓

$\operatorname{Arg}\theta = \pi/6,\ 5\pi/6,\ 3\pi/2$

1

$\sqrt{2}\left(\cos(\pi/6) + i\sin(\pi/6)\right)$

$\sqrt{2}\left(\cos(5\pi/6) + i\sin(5\pi/6)\right)$

$\sqrt{2}\left(\cos(3\pi/2) + i\sin(3\pi/2)\right)$

$\boxed{4}$

4) a) $\ddot{x} + 2\dot{x} + 2x = t^2 + 1$

$$x_p = at^2 + bt + c$$
$$\dot{x}_p = 2at + b$$
$$\ddot{x}_p = 2a$$

$$\boxed{x_p = \tfrac{1}{2}t^2 - t + 1}$$ ✓

$$(2a) + 2(2at + b) + 2(at^2 + bt + c) = t^2 + 1$$
$$2a + 4at + 2b + 2at^2 + 2bt + 2c = t^2 + 1$$

2

$$2at^2 = t^2 \qquad a = 1/2$$
$$4at + 2bt = 0 \qquad b = -1$$
$$2a + 2b + 2c = 1 \qquad 1 - 2 + 2c = 1$$
$$-1 \quad c = 1$$

b) $\ddot{x} + 2\dot{x} + 2x = e^{-2t} + 1$

$$\text{ERF} = \frac{e^{-2t}}{(-2)^2 + 2(-2) + 2} = \frac{e^{-2t}}{4 - 4 + 2} = \boxed{\frac{1}{2}e^{-2t} + \frac{1}{2} = x_p}$$

1

c) $\ddot{x} + 2\dot{x} + 2x = \sin t = \operatorname{Im}(e^{it})$

$$x_p = \operatorname{Im}\left(\frac{e^{it}}{(i)^2 + 2i + 2}\right) = \operatorname{Im}\left(\frac{e^{it}}{-1 + 2i + 2}\right) = \frac{\cos t + i\sin t}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i}$$

$$\frac{\cancel{\cos t} - 2i\cos t + i\sin t + \cancel{2\sin t}}{5}$$

$$y_p = \frac{1}{5}\sin(t) - \frac{2}{5}\cos(t)$$

$$\boxed{A = \frac{\sqrt{5}}{5}}$$ ✓

2

d) $x_p = t^3$

$x_p = at^3 + bt^2 + ct + d$

$\dot{x}_p = 3at^2 + 2bt + c$

$\ddot{x}_p = 6at + 2b$

$6at + 2b + 2(3at^2 + 2bt + c) + 2(at^3 + bt^2 + ct + d) = g(t)$

$a = 1, \quad b, c, d = 0$

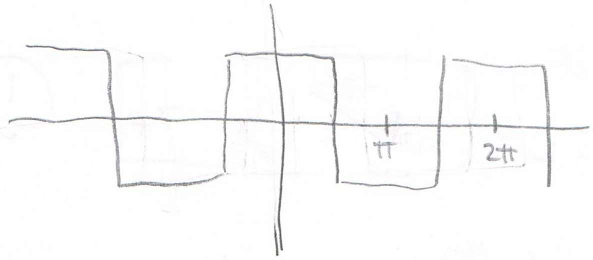
$\boxed{6t + 6t^2 + 2t^3 = g(t)}$ 2

e) $x = x_p + x_c$ $\quad \lambda^2 + 2\lambda + 2$

$x_c = \mathrm{Re}\left(c_1 e^{(-1+i)t}\right) + \mathrm{Im}\left(c_2 e^{(-1+i)t}\right) \quad \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$\boxed{x = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + t^3}$ 2 9

5) a)



3

b) $f = \frac{4}{\pi}\left(\sin(t + \pi/2) + \frac{\sin(3t + \pi/2)}{3} + \frac{\sin(5t + \pi/2)}{5} \cdots\right.$ 3

$f = \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} + \cdots\right.$ 9

c) should be cos (t)

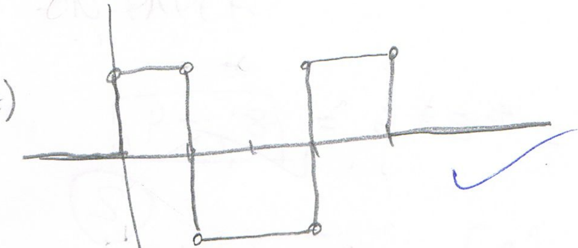
$x_p = \frac{4}{\pi}\left(\frac{t\,\cancel{\sin}(t)}{2} + \frac{\sin(3t)}{3(1-9)} + \frac{\sin(5t)}{5(1-25)} + \cdots\right.$ 2

$\frac{e^{-it}}{(i)^2 + 1} \Rightarrow \frac{-ite^{-it}}{2(i)} = \frac{\sin(nt)}{k - \omega n^2}$

first term is in resonance

6) a)

v(t)

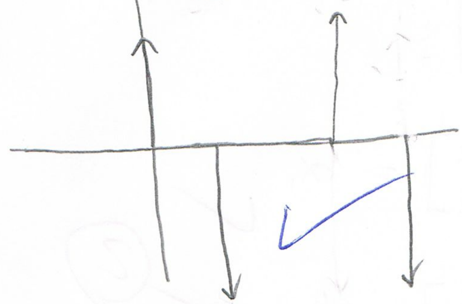


b) $u(t) - 2u(t-1) + 2u(t-3) - u($

3

i(t)

c)



d) $\dot{v}(t) = \delta(t) - 2(\delta(t-1) - \delta(t-3)) - $

3

$\boxed{7}$

e) $x = \int_0^t (u(t-\tau) - u(t-\tau-1))\, w(\tau)\, d\tau$ 1

$= \int_0^t w(\tau) \begin{cases} 1 \\ 0 \end{cases}$

$\cancel{a(t) = a}$
$\cancel{b(t) = 1}$

$\cancel{a(t) = t}$
$\cancel{b(t) = t+1}$

7) a) $W = \dfrac{1}{2s^2+8s+16}$ 3

b) $w = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+8s+16}\right\} = \dfrac{1}{2(s^2+4s+8)} = \dfrac{1}{2((s+2)^2+8)} \Rightarrow \dfrac{1}{2(s}$

$\dfrac{-4 \pm \sqrt{16-32}}{2} \quad -2 \pm i$

$= \dfrac{1}{2} \cdot \dfrac{1}{(s+2+i)(s+2-i)} = \dfrac{A}{(s+2+i)} + \dfrac{B}{(s+2-i)} = \dfrac{-1/4i}{(s+2+i)} + \dfrac{1/4i}{(s+2-}$ 2

A @ $s = -2-i$

$\dfrac{1/2}{-2i} = -1/4i$

B @ $s = -2+i$

$\dfrac{1/2}{2i} = 1/4i$

$\dfrac{-1}{4i} e^{(-2+i)t} + \dfrac{1}{4i} e$

c) $X = \dfrac{1}{2s^2+8s+16} \cdot \left(\dfrac{1}{s^2+1} + 2s + 12\right)$ 1

ALGEBRAIC ERROR −1

$+\frac{1}{4} e^{-2} \sin(t) + \frac{1}{4} e$

$\frac{1}{4} e^{-2t} \sin(2t)$

$\boxed{6}$

$X = WF$ [illegible]

8) a) $\lambda^2 - 4\lambda + (-32)$

$$\lambda = \frac{4 \pm \sqrt{16+4(32)}}{2} = 2 \pm 6 = \boxed{8, -4}$$

2

b) $\begin{bmatrix} 2-\lambda & 12 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-6a_1 + 12a_2 = 0$ $\quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \alpha_1$

$3a_1 - 6a_2 = 0$

$6a_1 + 12a_2 = 0$ $\quad \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \alpha_2$

$3a_1 + 6a_2 = 0$

2

c) $e^{Bt} = \Phi \Phi^{-1}$

$\Phi = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix}$ $\quad \Phi^{-1}(0)$

forgot to zero

X

1) $\begin{bmatrix} + & - \\ - & + \end{bmatrix} \Rightarrow \begin{bmatrix} e^t & e^{2t} \\ -e^t & e^{2t} \end{bmatrix}$

2) (T) $\begin{bmatrix} e^t & -e^t \\ e^{2t} & e^{2t} \end{bmatrix}$ 3) det $e^{3t} + e^{3t} = 2e^{3t}$

4) $\begin{bmatrix} \frac{e^{-2t}}{2} & -\frac{e^{-2t}}{2} \\ \frac{e^{-t}}{2} & \frac{e^{-t}}{2} \end{bmatrix}$

$$\Phi\Phi^{-1} = \boxed{\begin{bmatrix} \frac{e^{-t}}{2} - \frac{e^t}{2} & -\frac{e^{-t}}{2} - \frac{e^t}{2} \\ \frac{e^{-t}}{2} + \frac{e^t}{2} & -\frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix}}$$

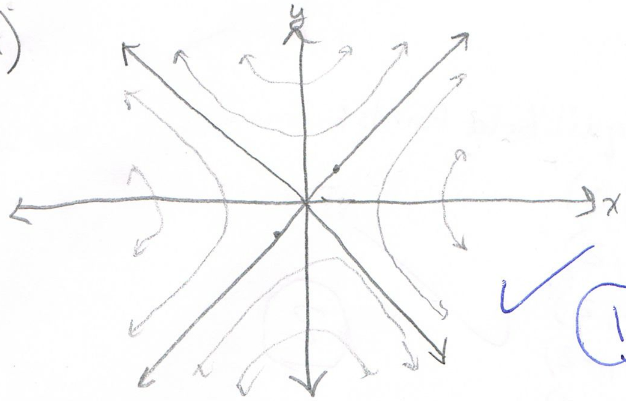
$\boxed{5}$

d) $u = \Phi u(0)$

$$= \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^t - e^{2t} \\ 2e^t + e^{2t} \end{bmatrix}$$ column vector (square)

X

9) a)



$x = c_1(1)e^t - c_2 e^{2t}$

$y = c_1 e^t + c_2 e^{2t}$

$c_1 \pm 1, \; c_2 = 0$

$c_1 = 0, \; c_2 = \pm 1$

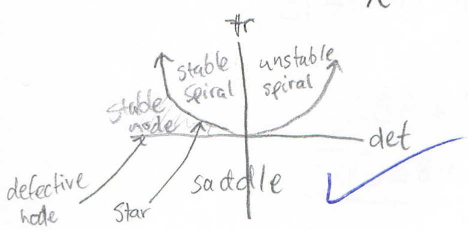
1

b) $A = \begin{pmatrix} a & -2 \\ 2 & 1 \end{pmatrix}$

$\lambda^2 - (a+1)\lambda + (a+\cancel{5})$

$$\lambda = \frac{(a+1) \pm \sqrt{(a+1)^2 - 4(a+5)}}{2}$$

0



i) Saddle when trace < 0

$a+1 < 0$

$\boxed{a < -1}$

ii) Star when $\text{trace}^2 = 4(\text{det})$

$(a+1)^2 = 4(a+5)$

$a^2 + 2a + 1 = 4a + 20$

$a^2 - 2a - 19 = 0$

$a = \frac{2 \pm \sqrt{4^2 - 4 \cdot 19}}{2}$

iii) ~~Stable Node when~~ ~~det(A)~~

~~$a > 1$ & $(a+1)^2 > 4(a+5)$ & $a < -5$~~

cannot occur in stated matrix

iv) Spiral (stable) when

$a < -5$ & $(a+1)^2 < 4(a+5)$

(det < 0) clockwise

v) Unstable spiral when

$a > -5$ & $(a+1)^2 < 4(a+5$

(det > 0) (complex eigenva

vi) Unstable defective node when

$a = -1$

1

10) a) $\dot{x} = x^2 - y^2$

$\dot{y} = x^2 + y^2 - 8$

$0 = x^2 - y^2$

$0 = x^2 + y^2 - 8$

$x^2 + y^2 = 8$

Equilibria Points:

$(0,0)$
$(2,2)$
$(-2,2)$
$(2,-2)$
$(-2,-2)$

2

b) SW Quadrant is $(-2,-2)$

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ 2x-8 & 2y-8 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ -12 & -12 \end{pmatrix}$$

0

$\lambda^2 + 48\lambda$

$\dfrac{-48 \pm \sqrt{48^2}}{}$

c) ω & ϕ are common

0

$\boxed{2}$

d) $\dot{x} = 2x - 3x^2 + x^3$

0

$\dot{x} = 2x$