Final Examination in Linear Algebra: 18.06

May 18, 1998 9:00–12:00 Professor Strang

Your name is: ______________________________

Grading

1	15
2	8
3	0
4	6
5	12
6	8
7	12
8	5
	66.1

Please circle your recitation:

1)	M2	2-132	M. Nevins	2-588	3-4110	monica@math
2)	M3	2-131	A. Voronov	2-246	3-3299	voronov@math
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4)	T12	2-132	A. Edelman	2-380	3-7770	edelman@math
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Answer all 8 questions on these pages. This is a closed book exam. Calculators are not needed in any way and therefore not allowed (to be fair to all). *Grades are known only to your recitation instructor.* Best wishes for the summer and thank you for taking 18.06.

GS

1 If A is a 5 by 4 matrix with linearly independent columns, find each of these **explicitly**:

(a) **(3 points)** The nullspace of A.

(b) **(3 points)** The dimension of the left nullspace $\boldsymbol{N}(A^T)$.

(c) **(3 points)** One particular solution x_p to $Ax_p =$ column 2 of A.

(d) **(3 points)** The general (complete) solution to $Ax =$ column 2 of A.

(e) **(3 points)** The reduced row echelon form R of A.

$\boxed{15}$

$$A = \begin{bmatrix} | & | & | & | \\ x_1 & x_2 & x_3 & x_4 \\ | & | & | & | \end{bmatrix}$$

a) $N(A) = \emptyset$ ← the matrix has rank 4, and only 4 unknowns, therefore only the zero vector is in the nullspace

b) $\dim(N(A^T)) = 1$ ← there is one dependent row giving a dimension of one for the leftnullspace

c) $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ Since all four columns are linearly independant the only way to get x_p is to contribute zero from all other columns and one times the second.

d) $x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ Since the nullspace is zero, there are no special solutions, only the particular solution

e) $R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ the first four rows $= I$ because they contain the pivot variables, the last row can be elimina[ted] as it is a linear combination of the other [rows]

2 (a) **(5 points)** Find the general (complete) solution to this equation $Ax = b$:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

(b) **(3 points)** Find a basis for the column space of the 3 by 9 block matrix $[A\ 2A\ A^2]$.

a) $$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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x_p: $x_1 + x_2 + 2x_3 = 2$, $-2x_3 = 0$

x_n: $x_1 + x_2 + 2x_3 =$, $-2x_3 =$

$x_3 = 0$, $x_1 = 1$, $x_2 = -1$

$$x_p = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad x_n = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$$

b) The colspace of A is defined by the pivot cols therefore: $C(A)$: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

A^2 is strictly a subspace of A because matrix multiplication is merely a linear transformation

2A has the same column space as A as it is also merely a linear transformation

Therefore a basis for $C([A\ 2A\ A^2])$: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

3 (a) **(5 points)** The command $N =$ **null** (A) produces a matrix whose columns are a basis for the nullspace of A. What matrix (describe its properties) is then produced by $B =$ **null** (N')? ← As this is a numeric matrix, I assume $N = N'$

(b) **(3 points)** What are the shapes (how many rows and columns) of those matrices N and B, if A is m by n of rank r?

a) N will have m rows and n−r cols

B will have n−r rows but only one column, the zero vector

$\boxed{0}$

$\boxed{B = \emptyset_{(n-r)\times 1}}$ X

b) $N = m \times (n-r)$

$B = (n-r) \times 1$ X

4 Find the determinants of these three matrices:

(a) **(2 points)**

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$\det(A) = -1 * (2 * (-3))$

$\boxed{= 6}$

(b) **(2 points)**

$$B = \begin{bmatrix} 0 & -A \\ I & -I \end{bmatrix} \quad \text{(8 by 8, same } A\text{)}$$

$\boxed{= 6}$

(c) **(2 points)**

$$C = \begin{bmatrix} A & -A \\ I & -I \end{bmatrix} \quad \text{(8 by 8, same } A\text{)}$$

$\boxed{= 0}$

$\boxed{6}$

5 **If possible** construct 3 by 3 matrices A, B, C, D with these properties:

(a) **(3 points)** A is a **symmetric** matrix. Its row space is spanned by the vector $(1, 1, 2)$ and its column space is spanned by the vector $(2, 2, 4)$.

(b) **(3 points)** All three of these equations have **no solution** but $B \neq 0$:

$$Bx = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Bx = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Bx = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(c) **(3 points)** C is a real square matrix but its eigenvalues are not all real and not all pure imaginary.

(d) **(3 points)** The vector $(1, 1, 1)$ is in the row space of D but the vector $(1, -1, 0)$ is not in the nullspace.

a) $\boxed{A = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 2 & 4 \\ 4 & 4 & 8 \end{bmatrix}}$ ✓

d) $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 7 & 9 & -4 \end{bmatrix}$ ✓

b) $B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ ✓

$\boxed{12}$

c) $\boxed{C = \begin{bmatrix} 1 & -5 \\ 3 & 1 \end{bmatrix}}$

$\lambda^2 - 2\lambda + 16$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{-60}}{2}$ ✓

$b^2 < 4ac$

$\text{trace}^2 < 4 \det(C)$

6 Suppose u_1, u_2, u_3 is an orthonormal basis for $\mathbf{R}^3$ and v_1, v_2 is an orthonormal basis for $\mathbf{R}^2$.

(a) **(5 points)** What is the **rank**, what are **all vectors** in the **column space**, and what is a **basis for the nullspace** for the matrix $B = u_1(v_1 + v_2)^T$?

(b) **(5 points)** Suppose $A = u_1 v_1^T + u_2 v_2^T$. Multiply AA^T and simplify. Show that this is a projection matrix by checking the required properties.

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(c) **(4 points)** Multiply $A^T A$ and simplify. This is the identity matrix! Prove this (for example compute $A^T A v_1$ and then finish the reasoning).

a) i) B has rank 1 because it ~~is~~ the product of two vectors.

3/5 ii) the colspace of B is spanned by u_1 therefore all vectors are cu_1 for $c \in \mathbb{R}$

iii) the nullspace of B is spanned by V_1 and V_2

b) $AA^T = (u_1v_1^T + u_2v_2^T)(u_1v_1^T + u_2v_2^T)^T$

$= (u_1v_1^T + u_2v_2^T)(v_1u_1^T + v_2u_2^T)$

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$(AA^T)^2 = (u_1v_1^T + u_2v_2^T)(\cancel{v_1u_1^T} + \cancel{v_2u_2^T})(\cancel{u_1v_1^T} + \cancel{u_2v_2^T})(v_1u_1^T + v_2u_2^T)$

$\therefore (AA^T)^2 = AA^T$ ← therefore AA^T is a projection matr

c) $A^TA = (\cancel{v_1u_1^T} + \cancel{v_2u_2^T})(\cancel{u_1v_1^T} + \cancel{u_2v_2^T}) = I$

0

$A^TAv_1 = (v_1u_1^T + v_2u_2^T)(u_1v_1^T + u_2v_2^T)v_1 = Iv_1 = v_1$

7 (a) **(4 points)** If these three points happen to lie on a line $y = C + Dt$, what system $Ax = b$ of three equations in two unknowns would be solvable?

4/4

$$y = 0 \text{ at } t = -1\,, \qquad y = 1 \text{ at } t = 0\,, \qquad y = B \text{ at } t = 1\,.$$

Which value of B puts the vector $b = (0, 1, B)$ into the column space of A?

4/4 (b) **(4 points)** For every B find the numbers $\overline{C}$ and $\overline{D}$ that give the best straight line $y = \overline{C} + \overline{D}t$ (closest to the three points in the least squares sense).

0/4 (c) **(4 points)** Find the projection of $b = (1, 0, 0)$ onto the column space of A.

2/2 (d) **(2 points)** If you apply the Gram-Schmidt procedure to this matrix A, what is the resulting matrix Q that has orthonormal columns?

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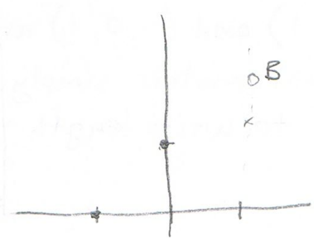
a) 

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \\ B \end{bmatrix}$$

if $B = 2$, this system is solvable

b) $A^TA\hat{x} = A^Tb$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ B \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 1+B \\ B \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \overline{C} \\ \overline{D} \end{bmatrix} = \begin{bmatrix} 1+B \\ B \end{bmatrix}$$

$3\overline{C} = 1 + B$

$2\overline{D} = B$

$$\boxed{\overline{D} = \frac{B}{2}, \quad \overline{C} = \frac{1+B}{3}}$$

CONT. ON BACK →

c) $b' = Pb$

$$A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$b' = A(A^T A)^{-1} A^T b$$

$$(A^T A)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$b' = \underset{3\times 2}{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$b' = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ X

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}}$$

d) $$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$ ✓

since $(1,1,1)$ and $(-1,0,1)$ are $\perp$ the orthonormal matrix simply scales these vectors to unit length

8 (a) **(5 points)** Find a complete set of eigenvalues and eigenvectors for the matrix

$\lambda = 1$

X $\alpha_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \alpha_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

$\det(A) = 2(3) - 1(1) + 1(-1)$
$= 6 - 1 - 1 = 4$

(b) **(6 points, 1 each)** Circle all the properties of this matrix A:

~~A is a projection matrix~~ ✓

A is a positive definite matrix ✓

~~A is a Markov matrix~~ ✓

~~A has determinant larger than trace~~ ✓

~~A has three orthonormal eigenvectors~~ X

A can be factored into $A = LU$ ✓

5/6

5/15

$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = [$

$A^2 \neq A$

(c) **(4 points)** Write the vector $u_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ as a combination of eigenvectors of A, and compute the vector $u_{100} = A^{100}u_0$.

a) $\det(A - \lambda I) = 0$

X $\lambda = 1$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\alpha = \emptyset$

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$\alpha_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\alpha_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\alpha_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

c) $\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$x_3 = -3$
$x_2 = -2$
$x_1 = 0$

$u_0 = -3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0+2 \\ -3-0 \\ 3 \end{bmatrix}$

CONT. ON BACK →

a) $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left((2-\lambda)(2-\lambda)-1\right) - 1\,(2-\lambda-1) + 1\,(1-2+\lambda)$

$-2+\lambda+1+1-2+\lambda$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1.5 & 0 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1.5 & 0 \\ 0 & 0.5 & 1.5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

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$$\begin{bmatrix} -7 & 1 & 1 \\ 1 & -7 & 1 \\ 1 & 1 & -7 \end{bmatrix} \begin{bmatrix} \\ \\ 1 \end{bmatrix}$$

$-7x_1 + x_2 = -1$

$x_1 - 7x_2 = -1$ $\quad -7 - x_2 - 7x_2 = -1$

$x_1 + x_2 = -7$ $\quad -8x_2 = 6$

$x_1 = -7 - x_2$ $\quad x_2 = -3/4$

$x_1 - 3/4 - 7 = 0$

$x_1 = 31/7$

$-7(31/7) - 3/4 + 1 = 0$

$-31 - 3/4 + 1 \cancel{= 0}$