LOGIC II - 24.242

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JUNE 4, 2012

HOMEWORK # 1

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1. Write down a bounded formula whose extension is the set of triples $<x, y, z>$ such that x, y and z are positive integers and z is a common divisor of x and y.

We want x, y and z such that:

$$(\exists a)\, z \cdot a = x \checkmark$$
$$(\exists b)\, z \cdot b = y \checkmark$$

replace this w/ non-negativity constraint

$\cancel{(\forall x < \gamma)}(\forall y < \gamma)(\forall z < \gamma)((\exists a <_s x)((z \cdot a) = x) \land (\exists b <_s y)$ ... $((z \cdot b) = y))$

use γ because question required a bounded formula.

Provided answer:

$(((0 < x \land 0 < y) \land 0 < z) \land \ldots$
$\ldots ((\exists u < sx)(u \cdot z) = x) \land (\exists v < sv)(v \cdot z) = y))$

*Note – variables can be free in the bounded formula!

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2. Define, for F, a finite set of the natural numbers, Code(F) to be $\sum_{x \in F} 2^x$, so that F is the set of places in the binary decimal expansion where 1s appear.

Give the Arabic numeral for Code({2, 4, 6, 8})

$$\text{Code}(\{2,4,6,8\}) = 2^2 + 2^4 + 2^6 + 2^8$$
$$4 + 16 + 64 + 256$$
$$= 340 \checkmark$$

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HOMEWORK #2

1. Write a program that calculates $(x+y)$ [using the register program model provided]

1. If the number in register 2 is zero, go to step 5.
2. Subtract 1 from register 2, unless that number is already zero.
3. Add 1 to the number in register 1.
4. Go to instruction 1.
5. STOP.

2. Show that a set is Δ iff its characterist function, χ_s, is Σ.

If a set is Δ, this means that we can effectively enumerate all members and non-members. This implies its characteristic function χ_s recursively enumerable because we can output "1" if [n] is an element of s and "0" if it is not an element, therefore χ_s is Σ

This proves S is $\Delta \Rightarrow \chi_s$ is Σ

If a function χ_s is Σ and a characteristic function, it must return "1" or "0" for all [n]. This produces the bounded formula $(_s0) = \chi_s([n])$. Since all bounded formulas extend to Δ this proves

χ_s is $\Sigma \Rightarrow S$ is Δ

Provided Answer:

($\Rightarrow$) If the set S is Δ, then there are bounded formulas $\varphi(x, y)$ and $\psi(x, y)$ such that $S = \{x : (\exists y)\varphi(x, y)\}$, and its complement $\{x : (\exists y)\psi(x, y)\}$. Then χ_S is equal to $\{\langle x, z\rangle : (\exists y)((\varphi([x], y) \wedge z = s$
$\dots \vee (\psi(x, y) \wedge z = 0))$

--- In basic terminology, this is stating that Δ sets have bounded formula for both positive cases and negative cases. By creating a function which maps to 1 in the positive case and 0 in the negative case, we get a Σ function ---

($\Leftarrow$) Suppose χ_S is Δ; say it's $\{\langle x, y\rangle : (\exists z)\theta($
$z)$

Then S is equal to $\{x : (\exists z)\theta([x], s0, z)$
and its complement $\{x : (\exists z)\theta([x], 0, z)$

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HOMEWORK #3

1. Give the Arabic numeral for $\ulcorner(0+0)\urcorner$

$\ulcorner(0+0)\urcorner$ $\quad\ulcorner 0\urcorner = 4$, not 0!

Pair(5, Pair($\ulcorner 0\urcorner$, $\ulcorner 0\urcorner$))

Pair(5, Pair(Pair(0,0), Pair(0,0)))

=

$\frac{1}{2}(0+0)(0+0+1)+0$

$= 0$ x $= 40$

Pair(5, 0)

$= \frac{1}{2}(5+0)(5+0+1)+5$

$= \boxed{20}$ x 1040

2. Show that a set of natural numbers is decidable iff it is either finite or the range of an increasing calculable total function.

($\Rightarrow$) If a set of numbers is finite, it is automatically decidable since both membership and non-membership can be determined simply through iteration.

If the set of numbers is infinite and the range of an increasing calculable total function, then we can decide membership placing y in S if $f([n]) = y$, counting upwards for n, and placing y in co-S if it is not in S and it is $< f([n])$.

($\Leftarrow$) A decidable set is the range of an increasing calculable function $f(x)$ where the domain $x \in \{0, 1, 2, 3\}$ matches one-to-one with the set S, with

$0 \rightarrow$ smallest element of S

$1 \longrightarrow 2^{nd}$ smallest element of S

$\vdots$

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HOMEWORK #4

1. Give a derivation from the empty set of premises, for $((A_0 \to A_1) \to ((A_1 \to A_2) \to (A_0 \to A_2)))$

Derivation Rules:

(PI) - Reiterate a premise/open assumption

(CP) - May write $(\phi \to \psi)$

(MP) - Given $(\phi \to \psi)$ and ϕ, write ψ

(MT) - Given $(\sim\phi \to \sim\psi)$ and ψ, write ϕ

PREMISE SET	SENTENCES
$\{A_2\}$	A_2
$\{A_2\}$	$(A_0 \to A_2)$

$\{1\}$	1. $(A_0 \to A_1)$	PI
$\{2\}$	2. $(A_1 \to A_2)$	PI
$\{3\}$	3. A_0	PI
$\{1, 3\}$	4. A_1	MP, 1, 2
$\{1, 2, 3\}$	5. A_2	MP, 2, 4
$\{1, 2\}$	6. $(A_0 \to A_2)$	CP, 3, 5
$\{1\}$	7. $((A_1 \to A_2) \to (A_0 \to A_2))$	CP, 2
$\emptyset$	8. ...	CP, 1, 7

2. Show that the set of codes of RSC sentences is Σ

By the earlier result that Pair(a, b) gives unique answers for every ⟨a, b⟩ ordered pair, we can unfold any code to its ⟨a, b⟩ pairing. If we do this recursively we will eventually reach either a 1, 2 or 3 as the a term for all nested Pair(a, b) terms, or a number which cannot be reduced, by the well-ordered principle and that unfolding pairs is a strictly decreasing operation. Once this is done we can assemble the RSC symbol and determine if it is a valid sentence, thereb allowing us to prove membership of that code in the set of all RSC codes. As a result, the set of codes is Σ.

The provided proof is in the form of a bounded formula proving Σ.

June 5, 201?

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$\boxed{\text{HOMEWORK \#5}}$

1. Show that "$(\forall x)(\forall y)(\forall z)(x E (y+z)) = ((x E y) \cdot (x E z$
is a theorem of Peano Arithmetic.

We can do this by using the induction axiom schema:

$$((R(0) \wedge (\forall x)(R(x) \rightarrow R(sx))) \rightarrow (\forall x)(R(x))$$

First we want the base case:

$$(\forall x)(\forall y)(x E (y+0)) = ((x E y) \cdot (x E 0)))$$

By Q3 $(y+0)$ becomes y
By Q7 $(x E 0)$ becomes $s0$
By #3 $((x E y) \cdot s0)$ becomes $(x E y)$
By the definition of $=$, $(x E y) = (x E y)$
is true for all x and y therefore the base case is true.

Next we need to prove the induction hypothesis namely:

$$(\forall x)(\forall y)(\forall z)[(x E (y+z) = ((x E y) \cdot (x E z))) \rightarrow (x E (y + sz) = ((x E y) \cdot (x E sz)))]$$

By Q4 we turn $x E (y+sz)$ into $x E s(y+z)$
By Q8 we turn it into $x E (y+z) \cdot x$
By Q8 we turn $x E sz$ into $x E z \cdot x$
By #5 we get:

$$(x E (y+z) \cdot x = (((x E y) \cdot (x E z)) \cdot x)$$

By the definition of $\cdot$, the hypothesis is true for all z, $\therefore$ the quoted statement is a theorem of PA.

Hilroy

2. Show that, where $\mathfrak{A}$ is a nonstandard model of PA, there isn't any formula $\phi(x)$ of the language arithmetic that is satisfied by all the standard numbers in $\mathfrak{A}$ but isn't satisfied by any of the non-standard elements

Allow a to be a non-standard number. If $\phi([n])$ is a formula satisfied by all standard numbers, then it must also be satisfied by a. To show this we examine all formula components, $=$, $<$ predicates and show if $(\forall x)\phi(x) \wedge (x_{st}$ that $\phi(a)$

$$= : \quad (\forall x)(x = y) \rightarrow (a = y) \text{ By } \forall E$$

$$< : \quad (\forall x)(x < y) \rightarrow (\exists b)(b < y)$$

↑ if $y = a$, then $sb = a$, b is both nonstandard and $< y$

The provided answer shows that the induction axiom fails if $\mathfrak{A}$ is true for all standard, but not all nonstandard, elements.

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HOMEWORK #6

1. Show that the following are equivalent for any set A:
   - A is recursively enumerable (aka Σ)
   - A is 1-reducible to the set of Gödel numbers for valid sentences
   - A is m-reducible to the set of Gödel numbers for valid sentences

Proof: We can define $f(n)$ as mapping the least member of A to the least Gödel number of a valid sentence, and the second leasts to each other, and so on. Because the Gödel numbers are Σ and by the well-ordered principle, such a reduction is possible if A is Σ. Since this f is a one-to-one function, A is both 1- and m-reducible to the Gödel numbers.

I missed the backwards derivation, that m-reducibility $\Rightarrow \Sigma$

2. Give an example of a Σ partial function which cannot be extended to a Σ total function

The partial function which takes Gödel numbers and returns their sentence cannot be extended because most Gödel numbers have no valid interpretati as a sentence. Inability to extend must cause a contradiction, or impossible set to occur.

Hilroy

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HOMEWORK #7

1. Take a sentence α so that α is provably equivalent to $(Bew_{PA}([\ulcorner\alpha\urcorner]) \vee Bew_{PA}([\ulcorner\sim\alpha\urcorner]))$. Is α decidable in PA? Is it true?

Bew_{PA} is the same as saying "there exists, following the axioms of Peano Arithmetic, a proof of the formula represented by the Gödel number $\ulcorner\alpha\urcorner$." α is therefore stating that it is the case that there either is a proof that it is true or a proof that it is false.

By the self-reference lemma:

$$PA \vdash (\alpha \leftrightarrow (Bew_{PA}([\ulcorner\alpha\urcorner]) \vee Bew_{PA}([\ulcorner\sim\alpha\urcorner])))$$

This tells us α is true iff either there is a proof α is true or there is a proof α is false. Since α cannot be true if there exists a proof of its negation, we can ignore the second half of the disjunction. Since $PA \vdash (\alpha \leftrightarrow Bew_{PA}([\ulcorner\alpha\urcorner]))$, it is both decidable and true.

Löb's Theorem

Hilroy

2. Show that, for each n, one can find an arithmetical formula τ_n such that, for each sentence ϕ,
$PA \vdash ([\ulcorner\phi\urcorner] < [n] \to (\tau_n([\ulcorner\phi\urcorner]) \leftrightarrow \phi))$

If τ_n is replaced with Bew_{PA}, then the following is true because it can be alternatively read as:

If the Gödel number for ϕ is less than n, then there is a proof of ϕ iff ϕ is true. ✓

The provided answer differed slightly using $\tau_n(x)$ to be the ~~dis~~junction of $(x = [\ulcorner\phi_n\urcorner] \wedge \phi_n)$ for all n.

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HOMEWORK #8

1. Show that a sentence is in KB iff it's valid for the class of frames $\langle W, R, I \rangle$, with R symmetric.

KB is the smallest set of formulas which contain:

(TC) – Every tautological consequence of KB is in KB.

(Nec) – If ϕ is in KB, so is $\Box\phi$

(K) – All instances of the schema $(\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi))$ are in KB

(B) – $(\Diamond\Box\phi \to \phi)$

A sentence which fits the schema (B) is valid for all symmetric frames because $\Diamond(\Box\phi)$ means that at least in at least one accessible world, ϕ is necessary (meaning ϕ is true for all worlds accessible from that world). Since R is symmetric, if we can access such a world, v, from a, so as to conclude $\Diamond(\Box\phi)$, then we must also conclude ϕ, because our world is accessible from v.

2. Prove de Jongh's theorem that all instances of the schema:

(4) $(\Box\phi \to \Box\Box\phi)$

are elements of the smallest normal modal system which include all instances of the schema:

(L) $(\Box(\Box\phi \to \phi) \to \Box\phi)$

$$(\Box(\Box(\psi \wedge \Box\psi) \to (\psi \wedge \Box\psi)) \to \Box(\psi \wedge \Box\psi))$$

If ψ is $\Box\phi$ then we get $\Box\Box\phi$ as one of the conjuncts of the conclusi

1. $((\phi \wedge \Box\phi) \to \phi)$ (TC)
2. $\Box((\phi \wedge \Box\phi) \to \phi)$ (Nec)
3. $(\Box((\phi \wedge \Box\phi) \to \phi) \to (\Box(\phi \wedge \Box\phi) \to \Box\phi))$ (K)
4. $(\Box(\phi \wedge$