24.244, Modal Logic

FINAL EXAM

You may use any notes on the website while taking this exam.

I. Propositional modal logic

All of the problems in part I are based on a modal logic devised by G. H. Von Wright called *Somewhere Else* (SE). In the intended interpretation of SE, '$\Diamond$p' means that p is true in some *other* possible world. So the accessibility relation is the *distinctness* relation: for all x,y $\in$ W, xRy iff $x \neq y$.

SE is the system K plus the following two axiom schemata:

$$\vdash \phi \supset \Box\Diamond\phi$$

$$\vdash \Diamond\Diamond\phi \supset (\phi \vee \Diamond\phi)$$

I. Prove that the following are theorems of the system SE:

1. $(q \wedge \Box p) \supset \Box\Box(p \vee q)$
2. $\Diamond(q \wedge \Box p) \supset \Box(p \vee q)$
3. $\Diamond p \supset \Box(p \vee \Diamond p)$
4. $(\Diamond p \wedge \Diamond{\sim}p) \supset \Diamond\Diamond p$

II. Show that each of the following sentences is *not* valid in the class of models in which R is the distinctness relation.

1. $\Box p \supset \Box\Box p$ COMPLETE
2. $\Box(\Box p \supset q) \vee \Box(\Box q \supset p)$
3. $p \supset \Diamond\Diamond p$
4. $(\Box\Diamond p \wedge \Box\Diamond{\sim}p) \supset \Diamond\Box\Diamond p$

III. Show that the class of frames that are *frames for the system SE* are the frames in which R is symmetric, and also meets the following condition

For all x,y,z $\in$ W, if xRy and xRz and $y \neq z$, then yRz.

Show that the frame of the canonical model for SE meets this condition.

IV. Assuming the completeness and soundness result for K, show that SE is characterized by the class of models in which R is the distinctness relation.

V. Show that any consistent sentence of SE is satisfied in some *finite* model in which the accessibility relation is the distinctness relation.

For each question, you may assume the results of any previous question (whether or not you succeed in getting those results.) Derivations may use, in addition to the primitive axiom schemata and rules, the following derived rules:

TF: If $(\psi_1 \supset (\ldots \supset (\psi_n \supset \phi)\ldots)$ is a tautology, and $\psi_1, \ldots \psi_n$ appear as previous lines in the proof, then one can infer ϕ.

Substitutivity of tautological equivalents: If $\psi_1 \equiv \psi_2$, is a tautology, and ϕ is a previous line of the proof, then one may infer the result of substituting ψ_1 for ψ_2 pin ϕ.

The K rule: If $\vdash (\psi_1 \supset (\ldots \supset (\psi_n \supset \phi)\ldots)$, then $\vdash (\Box\psi_1 \supset (\ldots \supset (\Box\psi_n \supset \Box\phi)\ldots)$

[illegible]

You may use any notes ... [illegible] ... while taking the exam.

1. Propositional modal logic

All of the problems [illegible] based on [illegible] system [illegible] by [illegible]. In the [illegible] SK [illegible] ... is true in some other possible world. So the accessibility relation [illegible] for all [illegible].

[illegible] is the system K plus the following two axiom schemata:

[illegible] ϕ

[illegible] ϕ [illegible] ϕ)

1. Prove that the following are theorems of the system SK:

[illegible]

[illegible]

2. Show that each of the following sentences is valid in the class of models in which R is [illegible] relation.

[illegible]

[illegible]

3. Show that the class of frames that are models for the system SK are those in which R is [illegible] and also satisfies the following condition:

For [illegible] w [illegible] and xR[illegible] then [illegible]

Show that the frame of the canonical model for SK meets this condition.

By ... [illegible] the completeness ... [illegible] that SK is characterized by the class of models in which R is the distinctness relation.

4. Show that any consistent sentence of SK is satisfied in some model in which the accessibility relation is the distinctness relation.

For each question, you may assume the results of any previous question (whether or not you succeeded in getting those results.) Derivations may use, in addition to the primitive axiom schemata and rules, the following derived rules:

PL: If [illegible] is a tautology, and [illegible] appear as previous lines in the proof, then one can infer ϕ.

Substitution of [illegible] equivalents: If [illegible] is a tautology, and ϕ is a previous line of the proof, then one may infer the result of substituting [illegible] for [illegible] in ϕ.

The K rule: [illegible]

II. Quantified modal logic

Terminology: (1) a standard model, is a model as defined in the classical semantics, as contrasted with the counterpart semantics. (2) By an S5 model, I mean a model with a universal accessibility relation. (3) By quantified K, or quantified S5, I mean the logic obtained by combining the propositional modal logic K, or S5, with the extensional free predicate logic with identity. This is the logic that is complete for the counterpart semantics. (4) To say that something has a property essentially is to say that it has that property in all accessible possible worlds in which that thing exists

A. For each of the following, construct a standard S5 model to show that it is not valid:

1. $\Box s=t \supset \exists(x=t \wedge \Box x=s)$
2. $\sim Rxy(a)(b) \equiv \sim Rxy(a)(b)$
3. $\exists\Box(Ex \supset x=t) \supset \exists\Box(Et \supset x=t)$
4. $\exists\sim(\Box Fx \wedge \Box Gx) \supset \Box\Diamond\exists\sim(Fx \wedge Gx)$
5. $\forall[(Fx \wedge \Box\forall(Fy \supset x=y)) \supset \Box(Ey \supset Fy)(x)]$

B. Formalize the following in a way that reveals the relevant modal structure:

1. There might have been unicorns, but there is nothing that might have been a unicorn.
2. There is a man that Hillary might have married, but didn't.
3. It is essential to GWB that GHWB be his father, but it is not essential to GHWB that he be GWB's father. COMPLETE
4. God could make a stone that he could not lift.
5. Only one candidate can win.

C. In S4 or S5, we can say that a name 't' is a rigid designator with a single sentence, RD:

$\Box((Ex \vee Et) \supset x=t)(t)$

The following sentence, ERD, says that 't' is *essentially* a rigid designator:

$\Box(Et \rightarrow \Box((Ex \vee Et) \supset x=t)(t))$.

Show that there is a counterpart model in which a name 't' is a rigid designator, relative to a given world, but not an essentially rigid designator, relative to that world (that is, define a model in which RD is true, but ERD is false.)

D. Show that the following is a derived rule of quantified S5:

If $\vdash \psi \supset \Box\phi$, and there are no free occurrences of 'x' in ψ, then $\vdash \psi \supset \Box\forall\phi$ (Strictly, as the system is defined, only closed sentences are theorems, but for convenience, we can treat variables as names when they occur free. So in this case, 'x' may be free in ϕ.

E. Consider the claim that there might have existed things that do not in fact exist. It is not possible to express this claim in the quantified modal language we have been using, but there are expressible statements that entail it. Give an example of one. Then consider how the language might be enriched or modified to make the claim expressible.

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Scott Young
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I) I) Prove the following theorems:

1) $(q \wedge \Box p) \supset \Box\Box(p \vee q)$

(2)	1.	$\Diamond\Diamond(q \wedge \Box p) \supset ((q \wedge \Box p) \vee \Diamond(q \wedge \Box p))$
(K)	2.	$\Box\Diamond\Diamond(q \wedge \Box p) \supset \Box((q \wedge \Box p) \vee \Diamond(q \wedge \Box p))$
(1)	3.	$\phi \supset \Box\Diamond\phi$
[3](us)	4.	$\Diamond(q \wedge \Box p) \supset \Box\Diamond\Diamond(q \wedge \Box p)$
[2,4](T)	5.	$\Diamond(q \wedge \Box p) \supset \Box((q \wedge \Box p) \vee \Diamond(q \wedge \Box p))$
repeat 2-5 again	6.	$(q \wedge \Box p) \supset \Box\Box((q \wedge \Box p) \vee \Diamond(q \wedge \Box p))$
[6](T)	7.	$(q \wedge \Box p) \supset \Box\Box(q \vee \Diamond\Box p)$
(T)(1)	8.	$\sim\Box\Diamond\phi \supset \sim\phi$
[8](4n)?	9.	$\Diamond\Box\sim\psi \supset \sim\Box\Diamond\psi$
[8,9](T)	10.	$\Diamond\Box p \supset p$
	11.	$(q \wedge \Box p) \supset \Box\Box(q \vee \Diamond\Box p)$

Q.E.D.

✓

2) $\Diamond(q \wedge \Box p) \supset \Box(p \vee q)$

(2)	1.	$\Diamond\Diamond(q \wedge \Box p) \supset ((q \wedge \Box p) \vee \Diamond(q \wedge \Box p))$
previous result	2.	$\Diamond(q \wedge \Box p) \supset \Box((q \wedge \Box p) \vee \Diamond(q \wedge \Box p))$
[2] (T)	3.	$\Diamond(q \wedge \Box p) \supset \Box(q \vee \Diamond\Box p)$
previous result	4.	$\Diamond(q \wedge \Box p) \supset \Box(q \vee p)$

Q.E.D.

✓

3) $\Diamond p \supset \Box(p \vee \Diamond p)$

(2)	1.	$\Diamond\Diamond\phi \supset (\phi \vee \Diamond\phi)$
(K)	2.	$\Box\Diamond\Diamond\phi \supset \Box(\phi \vee \Diamond\phi)$
(1)	3.	$\phi \supset \Box\Diamond\phi$
[3](US)	4.	$\Diamond\phi \supset \Box\Diamond\Diamond\phi$
[2,4](T)	5.	$\Diamond\phi \supset \Box(\phi \vee \Diamond\phi)$
[5](US)	6.	$\Diamond p \supset \Box(p \vee \Diamond p)$
		Q.E.D.

4) $(\Diamond p \wedge \Diamond \sim p) \supset \Diamond\Diamond p$

1.	$\Diamond p \supset \Box(p \vee \Diamond p)$
2.	$\Diamond p \supset \Box(\sim\Diamond p \supset p)$
3.	$\Diamond p \supset (\Box\sim\Diamond p \supset \Box p)$
4.	$(\Diamond p \wedge \sim\Box p) \supset \Diamond\Diamond p$

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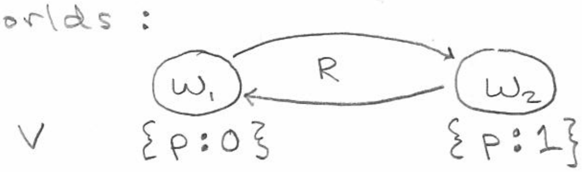
I) II) Show the following are not valid in the class of models where R is the distinctness relation.

1) $\Box p \supset \Box\Box p$

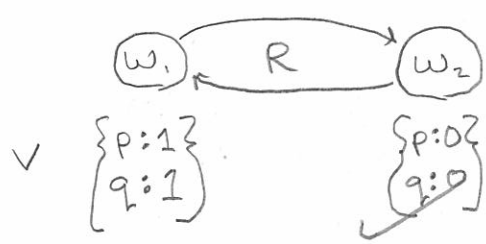
To show a given wff is not valid for a class of models we must simply construct a model where it is false. for this we will consider the model consisting of two worlds:



Here $\Box p \supset \Box\Box p$ is false from w_1 because $\Box p$ is true as w_2 is the only accessible world from w_1 and p is true in w_2, however $\Box\Box p$ is false as $\Box p$ is false in w_2 as p is false in w_1

2) $\Box(\Box p \supset q) \vee \Box(\Box q \supset p)$

Again we choose a two-world model with the following R and V:



Here $\Box p$ and $\Box q$ are true from w_2 but p and q are false, therefore, from w_1, $\Box(\Box p \supset q)$ is false and $\Box(\Box q \supset p)$ is false so the entire formula is false.

3) $p \supset \Diamond\Diamond p$

Here we need only consider the dead-end world where p is true. $\Diamond\Diamond p$ is false for any dead-end, so the formula is false for this model

4) $(\Box\Diamond p \wedge \Box\Diamond{\sim}p) \supset \Diamond\Box\Diamond p$

Again this is invalid for dead-end worlds as both $\Box\Diamond p$ and $\Box\Diamond{\sim}p$ are vacuously true while $\Diamond\Box\Diamond p$ is false for arbitrary V.

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III) Show that the class of frames that are frames for the system SE are the frames in which R is symmetric and also meets the following condition:

(*) For all $x, y, z \in W$, if xRy and xRz and $y \neq z$ then yRz

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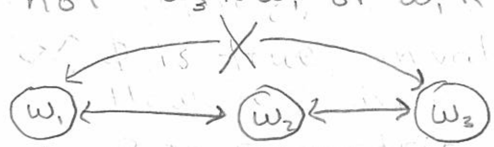
Axiom 1; $\vdash \phi \supset \Box\Diamond\phi$ requires R to be symmetric since if there existed worlds ω_1 and ω_2 such that $\omega_1 R \omega_2$ but not $\omega_2 R \omega_1$, then if $\phi : 1$ for ω_1 and $\phi : 0$ for all other worlds then $\phi \supset \Box\Diamond\phi$ would be false, contradicting this as an axiom valid in all frames.

Axiom 2, $\vdash \Diamond\Diamond\phi \supset (\phi \vee \Diamond\phi)$ requires R to adhere to property (*) since if there existed a model where xRy and xRz and $y \neq z$ but not yRz, we could construct a symmetrical model such that $\omega_1 R \omega_2$; $\omega_2 R \omega_3$, $\omega_3 R \omega_2$, $\omega_2 R \omega_1$ but not $\omega_3 R \omega_1$ or $\omega_1 R \omega_3$ like so



Then, with $\phi : 0$ for ω_1 & ω_2 and $\phi : 1$ for ω_3, we have $\Diamond\Diamond\phi$ true, yet both ϕ and $\Diamond\phi$ false, contradicting the axiom.

Show that the frame of the canonical model meets this condition.

The canonical model consists of $\omega \in W$ iff ω is a maximally SE-consistent set of wff and $V_\omega \rightarrow \{\alpha : 1\}$ iff $\alpha \in \omega$. R is defined as $\omega_1 R \omega_2$ iff $L^-(\omega_1) \subseteq \omega_2$, so our task is to show this meets the above conditions.

1) Symmetry. We know this is preserved because if $\alpha \in \omega$ then $\Box\Diamond\alpha \in \omega$ by axiom 1 and therefore $\Diamond\alpha \in$ of all ω' s.t. $\omega R \omega'$, however the only way for this to be true is if $\alpha \in \omega^*$ s.t. $\omega' R \omega^*$, which can only be true for all α if R is Symmetric (that is $\omega^* = \omega$).

III) CONT.)

2) Property (*). We know this is preserved because if $\Diamond\Diamond\alpha \in w$ then so must be either $\alpha \in w$ or $\Diamond\alpha \in w$. Were this not the case then there could be MCC sets of sentences w_1, w_2, w_3 s.t. $w_1 R w_2$, $w_2 R w_3$ but not $w_1 R w_3$ as such α or $\Diamond\alpha$ may not necessarily $\in w$.

IV) Assuming the soundness and completeness of K, show that SE is characterized by the class of models where R is the distinctness relation.

Showing R characterizes SE amounts to proving that SE is sound and complete relative to the class of models with R. We have shown by III that SE is sound relative to the class of models where R is the distinctness relation, now we must prove it is complete.

To do this we will create the canonical model for SE and show that if α is valid (w.r.t. class $\mathcal{E}$ w/ R) then $\vdash_{SE} \alpha$. We already know from III that the frame of this model is in $\mathcal{E}$, so now we only need show that any SE-consistent set of sentences, Γ_{SE}, is satisfiable.

X

We can prove completeness by induction on the length of α. The base case where α is a sentence letter is trivially true since if $\alpha \in \Gamma$ then $V(\alpha) = 1$ in Γ.

Next we show with α of the form $\sim\phi$:

1. $V(\alpha) = W - V(\psi)$
2. $W - V(\psi) = W - \{\Delta \in W ; \psi \in \Delta\}$
3. $W - \{\Delta \in W : \psi \in \Delta\} = \{\Delta \in W : \alpha \in \Delta\}$

Now we continue with α of the form $(\psi_1 \wedge \psi_2)$:

1) $v(\alpha) = v(\psi_1) \cap v(\psi_2)$
2) $v(\psi_1) \cap v(\psi_2) = \{\Delta \in W : \psi \in \Delta\} \cap \{\Delta \in W : \psi \in \Delta\}$
3) $\{\Delta \in W : \psi \in \Delta\} \cap \{\Delta \in W : \psi \in \Delta\} = \{\Delta \in W : \alpha \in \Delta\}$

Finally we look at the form α as $\Box\psi$:

1) $v(\alpha) = \{\Delta : \{\Theta : \Delta R \Theta\}\} \subseteq v(\psi)$
2) $v(\psi) = \{\Theta : \psi \in \Theta\}$
3) $v(\alpha) = \{\Delta : \{\Theta : \Delta R \Theta\}\} \subseteq \{\Theta : \psi \in \Theta\}$
4) $\psi \in \Theta$ for all Θ such that $L^{-}(\Delta) \subseteq \Theta$ iff $\alpha \in \Delta$
5) $v(\alpha) = \{\Delta \in W : \phi \in \Delta\}$

Since we have already shown the canonical model frame has the distinctness relation for R, this shows that SE is characterized by such frames.

V) Show that SE possesses the finite model property

To do this we construct a model using only the well-formed proper parts of any sentence $\underline{a}$. We then allow worlds to be a-maximally consistent sets of sentences, or finite sets which if $\beta \in \Phi_a$ (Φ_a is the set of all wf parts of a) then $\beta \in \Gamma$ or $\sim\beta \in \Gamma$, subject to the normal rules of consistency. W is the set of all such a-maximally consistent sets. We show completeness by induction on the length of α as before (which follows nearly identically, so I won't repeat it here), this guarantees that $V(\beta) = 1$ iff $\beta \in w$. Now the only remaining step is to show the frame $\langle W_a, R_a \rangle$ satisfies the class $\mathcal{E}$ of models. For this we repeat the analysis of III by showing:

1) R_a is symmetric, as if $\beta \in w$ then so is $\Box\Diamond\beta$. this is true because otherwise axiom 1 would be contradicted
2) Property (*), follows the same reasoning but with the second axiom.

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A) Construct a standard S5 model to show it is invalid

1) $\Box s = t \supset \exists(x = t \wedge \Box x = s)$

Here we construct a model with two worlds, w_1 and w_2. s and t refer to the same object, but a different one in each world. Therefore $\Box s = t$ is true, but $\exists(x = t \wedge \Box x = s)$ is false because there is no object which is equal to t in one world, yet equal to s in every world.

✓

2) $\sim Rxy(a)(b) \equiv \sim Rxy(a)(b)$

If a and b are free variables, they could have different assignments, so all we need is a world where $Rxy(a)(b)$ is neither truth-functionally true or false for all objects.

✗

3) $\exists\Box(Ex \supset x = t) \supset \exists\Box(Et \supset x = t)$

Here we construct a model

✗

4) $\exists \sim (\Box Fx \wedge \Box Gx) \supset \Box \Diamond \exists \sim (Fx \wedge Gx)$

Consider a model with two worlds, w_1 and w_2. In these worlds the domain consists of only one object, which is different in every world but is always true of both Fx and Gx. Therefore $\Box Fx$ and $\Box Gx$ is false always because there is no object that even exists for all worlds therefore $\exists \sim (\Box Fx \wedge \Box Gx)$ is true. However $\Box \Diamond \exists \sim (Fx \wedge Gx)$ is false because in every world there is only one thing and, for this thing, Fx and Gx are true. ✓

5) $\forall [(Fx \wedge \Box \forall (Fy \supset x = y)) \supset \Box (Ey \supset Fy)(x)]$

For this we need only consider a model where Fy is false for some world, but not this one. Then the conjunction $Fx \wedge \Box \forall (Fy \supset x = y)$ will be true since, for all objects in this world, Fx, and for all worlds <u>if</u> Fx (for all x) then $x = y$, however it is not the case, that, in all worlds if y exists then Fy because there is at least one world where it is not true. ✓

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B) 1) There might have been unicorns, but there is nothing that might have been a unicorn.

Let Ux mean "x is a unicorn."

$$\Diamond \exists x (Ux) \wedge \neg \exists x (\Diamond Ux)$$ ✓

2) There is a man Hillary might have married, but didn't.

Let Mx mean "x is a man Hillary married"

$$\exists x (\Diamond Mx \wedge \neg Mx)$$ ✓

3) It is essential to GWB that GHWB be his father, but it is not essential to GHWB to be GWB's father.

Let Fxy mean "x is y's father"

and g mean GWB

g' mean GHWB

$$\Box (Eg \supset Fg'g) \wedge \neg \Box (Eg' \supset Fg'g)$$ ✓

4) God could make a stone that he could not lift.

Let Mx mean "God made x."

Sx mean "x is a stone."

Lx mean "God ~~can~~ lift x."

← could not should imply over all possible worlds!

$$\Diamond \exists x ((Mx \wedge Sx) \wedge \neg Lx)$$

5) Only one candidate can win.

Let Wx mean "x is a candidate who won."

$$\Box \forall x (Wx \supset \forall y (Wy \supset x = y))$$

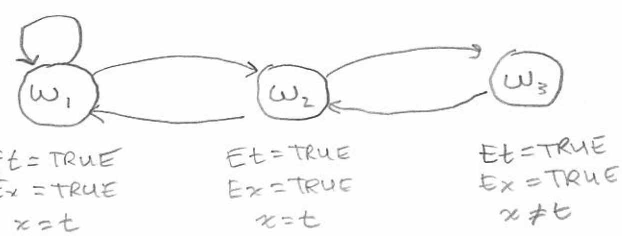
c) Show that there is a counterpart model in which a name, t, is a rigid designator but not essentially so.

RD: $\Box((Ex \vee Et) \supset x = t)$

ERD: $\Box(Et \supset \Box((Ex \vee Et) \supset x = t))$

This can be shown by providing a model, $\langle W, R, D, V \rangle$ in which RD is true, but ERD is false.

First, RD is true. This means $(Ex \vee Et) \supset x = t$ is true in all worlds accessible to @. ERD is false which means in at least one world, Et is true but t is not a rigid designator relative to that world. Consider, then, the following model:



1/2 (forgot to write as counterpart semantics)

From the perspective of world w_1, t is a rigid designator because RD is true. ERD, however is not true because RD is not true in w_2 although Et is.

D) Show the following is a derived rule of quantifies S5:

If $\vdash \psi \supset \Box \phi$ and there are no free occurrences of 'x' in ψ then $\vdash \psi \supset \Box \forall \phi$

First we use the rule of universal generalization to transform $\vdash \psi \supset \Box \phi \Rightarrow \vdash \psi \supset \forall \Box \phi$. Then, by the Barcan formula $\forall \Box \phi \supset \Box \forall \phi$ so $\vdash \psi \supset \Box \forall \phi$.

1 $\psi \supset \Box \phi$
2 $\psi \supset \forall \Box \phi$
3 | ψ
4 | $\forall \Box \phi$
5 | $\forall \Box \phi \supset \Box \forall \phi$
6 | $\Box \forall \phi$
7 $\psi \supset \Box \forall \phi$

Q.E.D.

X

E) $\Diamond Ex \wedge \neg Ex$ entails that there are things (in this case, x) which might have existed (in some other world), but which don't exist.

The language could be possibly extended by allowing a defined existence predicate E which maps objects within worlds to true if $a \in D_w$ and false if otherwise. Then all domains would be universal, but the existence of an object would be determined by E.

Evaluation:

Since no marking rubric was included, I've simply counted each question & subquestion as one mark:

Total 17/24 = 70.8%