

FINAL EXAM

You may use any notes on the website while taking this exam.

I. Propositional modal logic

All of the problems in part I are based on a modal logic devised by G. H. Von Wright called *Somewhere Else* (SE). In the intended interpretation of SE, ' $\Diamond p$ ' means that  $p$  is true in some *other* possible world. So the accessibility relation is the *distinctness* relation: for all  $x, y \in W$ ,  $xRy$  iff  $x \neq y$ .

SE is the system K plus the following two axiom schemata:

$$\vdash \phi \Rightarrow \Box \Diamond \phi$$

$$\vdash \Diamond \Diamond \phi \Rightarrow (\phi \vee \Diamond \phi)$$

I. Prove that the following are theorems of the system SE:

$$1. \Diamond (q \vee p) \Rightarrow \Box \Box (p \vee q)$$

$$2. \Diamond (q \wedge \Box p) \Rightarrow \Box (p \vee q)$$

$$3. \Diamond p \Rightarrow \Box (p \vee \Diamond p)$$

$$4. (\Diamond p \vee \Diamond \sim p) \Rightarrow \Diamond \Diamond p$$

II. Show that each of the following sentences is *not* valid in the class of models in which R is the distinctness relation.

COMPLETE

$$1. \Box p \Rightarrow \Box \Box p$$

$$2. \Box (\Box p \Rightarrow q) \vee \Box (\Box q \Rightarrow p)$$

$$3. p \Rightarrow \Diamond \Diamond p$$

$$4. (\Box \Diamond p \vee \Box \Diamond \sim p) \Rightarrow \Box \Box \Diamond p$$

III. Show that the class of frames that are *frames for the system SE* are the frames in which R is symmetric, and also meets the following condition

For all  $x, y, z \in W$ , if  $xRy$  and  $xRz$  and  $y \neq z$ , then  $yRz$ .

Show that the frame of the canonical model for SE meets this condition.

IV. Assuming the completeness and soundness result for K, show that SE is characterized by the class of models in which R is the distinctness relation.

V. Show that any consistent sentence of SE is satisfied in some *finite* model in which the accessibility relation is the distinctness relation.

For each question, you may assume the results of any previous question (whether or not you succeed in getting those results.) Derivations may use, in addition to the primitive axiom schemata and rules, the following derived rules:

TF: If  $(\psi_1 \Rightarrow (\psi_n \Rightarrow (\dots \Rightarrow (\psi_n \Rightarrow \phi))))$  is a tautology, and  $\psi_1, \dots, \psi_n$  appear as previous lines in the proof, then one can infer  $\phi$ .

Substitutivity of tautological equivalents: If  $\psi_1 \equiv \psi_2$  is a tautology, and  $\phi$  is a previous line of the proof, then one may infer the result of substituting  $\psi_1$  for  $\psi_2$  in  $\phi$ .

The K rule: If  $\vdash (\psi_1 \Rightarrow (\psi_2 \Rightarrow (\dots \Rightarrow (\psi_n \Rightarrow \phi))))$ , then  $\vdash (\Box \psi_1 \Rightarrow (\Box \psi_2 \Rightarrow (\dots \Rightarrow (\Box \psi_n \Rightarrow \Box \phi))))$ .



## II. Quantified modal logic

Terminology: (1) a standard model, is a model as defined in the classical semantics, as contrasted with the counterpart semantics. (2) By an S5 model, I mean a model with a universal accessibility relation. (3) By quantified K, or quantified S5, I mean the logic obtained by combining the propositional modal logic K, or S5, with the extensional free predicate logic with identity. This is the logic that is complete for the counterpart semantics. (4) To say that something has a property essentially is to say that it has that property in all accessible possible worlds in which that thing exists

A. For each of the following, construct a standard S5 model to show that it is not valid:

$$1. \Box s=t \supset \exists (x=t \wedge \Box x=s)$$

$$2. \sim Rxy(a)(b) \equiv \sim Rxy(a)(b)$$

$$3. \exists \Box (Ex \supset x=t) \supset \exists \Box (Et \supset x=t)$$

$$4. \exists \sim (\Box Fx \wedge \Box Gx) \supset \Box \exists \sim (Fx \wedge Gx)$$

$$5. A[(Fx \wedge \Box A(Fy \supset x=y)) \supset \Box (Ey \supset Fy)](x)$$

B. Formalize the following in a way that reveals the relevant modal structure:

1. There might have been unicorns, but there is nothing that might have been a unicorn.

2. There is a man that Hillary might have married, but didn't.

3. It is essential to GWB that GHWB be his father, but it is not essential to GHWB that he be

GWB's father.

4. God could make a stone that he could not lift.

5. Only one candidate can win.

C. In S4 or S5, we can say that a name 't' is a rigid designator with a single sentence, RD:

$$\Box ((Ex \vee Et) \supset x=t)(t)$$

The following sentence, ERD, says that 't' is *essentially* a rigid designator:

$$\Box (Et \supset \Box ((Ex \vee Et) \supset x=t)(t)).$$

Show that there is a counterpart model in which a name 't' is a rigid designator, relative to a given world, but not an essentially rigid designator, relative to that world (that is, define a model in which RD is true, but ERD is false.)

D. Show that the following is a derived rule of quantified S5:

If  $\vdash \psi \supset \Box \phi$ , and there are no free occurrences of 'x' in  $\psi$ , then  $\vdash \psi \supset \Box A\phi$  (Strictly, as the

system is defined, only closed sentences are theorems, but for convenience, we can treat variables as

names when they occur free. So in this case, 'x' may be free in  $\phi$ .

E. Consider the claim that there might have existed things that do not in fact exist. It is not possible to

express this claim in the quantified modal language we have been using, but there are expressible

statements that entail it. Give an example of one. Then consider how the language might be enriched or modified to make the claim expressible.



I) Prove the following theorems:

$$1) (q \vee r) \supset \Box(q \vee r)$$

- 2)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 3)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 4)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 5)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 6)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 7)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 8)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 9)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 10)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 11)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$

$\Box, \exists, \forall$

$$11) (q \vee r) \supset \Box(q \vee r)$$

- 1)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 2)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 3)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
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- 11)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$

$$2) (q \vee r) \supset \Box(q \vee r)$$

- 1)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 2)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 3)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$
- 4)  $\Box(p \supset q) \supset \Box(p \supset \Box q)$

$\Box, \exists, \forall$

$$3) \quad \Diamond p \supset \Box(p \vee \Diamond p)$$

$$(2) \quad 1. \quad \Diamond \Diamond \phi \supset (\phi \vee \Diamond \phi)$$

$$(K) \quad 2. \quad \Box \Diamond \Diamond \phi \supset \Box(\phi \vee \Diamond \phi)$$

$$(1) \quad 3. \quad \phi \supset \Box \Diamond \phi$$

$$[3](us) \quad 4. \quad \Diamond \phi \supset \Box \Diamond \Diamond \phi$$

$$[2,4](\tau) \quad 5. \quad \Diamond \phi \supset \Box(\phi \vee \Diamond \phi)$$

$$[5](us) \quad 6. \quad \Diamond p \supset \Box(p \vee \Diamond p)$$

Q.E.D.



$$4) \quad (\Diamond p \wedge \Diamond \sim p) \supset \Diamond \Diamond p$$

$$(1) \quad 1. \quad \Diamond p \supset \Box(p \vee \Diamond p)$$

$$(1) \quad 2. \quad \Diamond p \supset \Box(\sim \Diamond p \supset p)$$

$$3. \quad \Diamond p \supset (\Box \sim \Diamond p \supset \Box p)$$

$$4. \quad (\Diamond p \wedge \sim \Box p) \supset \Diamond \Diamond p$$

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