6.003 (Fall 2007)

Final exam

17 December 2007

Name: Scott Young

Please circle your section number:

Section	*Instructor*	*Time*
1	Jeffrey Lang	10
2	Jeffrey Lang	11
3	Karen Livescu	11
4	Sanjoy Mahajan	12
5	Antonio Torralba	1
6	Qing Hu	2

Partial credit *will* be given, according to the conceptual features that a proposed answer shares with the correct answer.

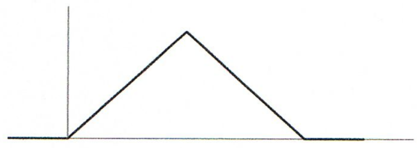
Explanations are not required and do not affect your grade.

- **You have three hours. Have fun!**
- Please put your initials on all subsequent sheets.
- Enter your answers in the boxes.
- This quiz is closed book, but you may use three 8.5 × 11 sheets of paper (six sides).
- No calculators, computers, cell phones, music players, or other aids.

1. 5/15 (5)	5. 4/05 (4)	9. 10/10 ( )	/ 30 ( )
2. 5/05 (5)	6. 5/10 (5)	10. 5/10 ( )	/ 25 ( )
3. 5/05 (5)	7. 12.5/15 (12.5)	11. 10/10 ( ) ☐	/ 30 ( )
4. 2.5/10 (2.5)	8. 5/05 (5)		/ 15 ( )
17.5/35 ( )	26.5/35 ( )	25/30 ( )	69 /**100** ( )

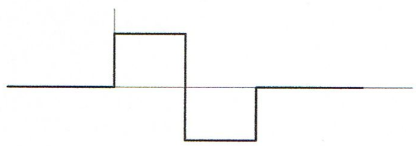
1. Matching time and frequency representations *[15 points]*

For each time signal, choose the *magnitude* of its Fourier transform from the four choices below. The time signals are zero outside the plotted region. Each time and frequency figure has its own scale, with the origin where the axes intersect. *If you are unsure of the correct answer, you can also select a second answer, in which case you will receive the average of the scores for the two answers.*



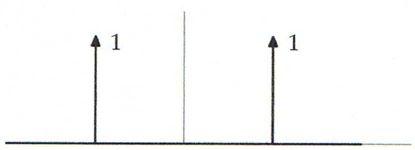
A B C D

5/15



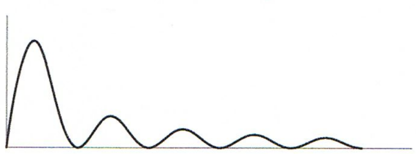
CTFT should be similar to sinc function

A B C D



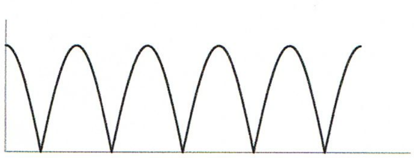
A B C D

Here are the choices for the Fourier transform magnitude:

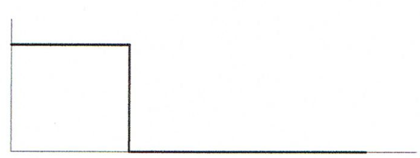


A

B



C



D

2. Discrete-time periodicity *[5 points]*

Here are two periodic discrete-time signals:

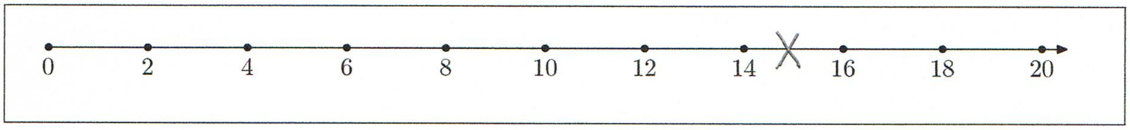
$$x_3[n] = \ldots, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$$ Every 3rd

$$x_5[n] = \ldots, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots$$ Every 5th

LCD = 15

Mark the period of their sum $x_3[n] + x_5[n]$ as an **X** on the thermometer:

✓



If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.

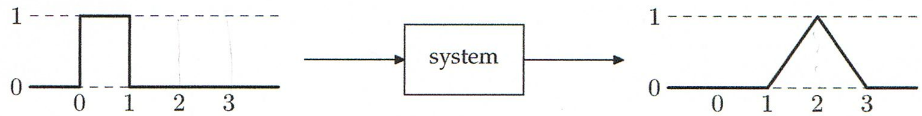
$(x_3 + x_5)[n] =$

5/5

0, 0, 1, 0, 1, 1, 0, 0, 1

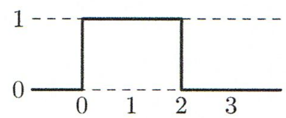
3. Find the output signal *[5 points]*

A given linear, time-invariant system turns the unit pulse into the triangle:

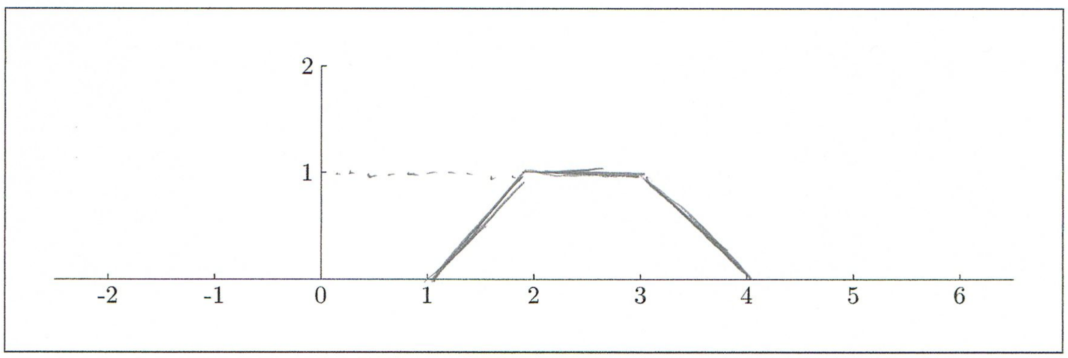


The system is given the following input signal:

h(System) =

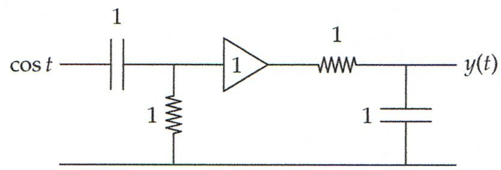


Sketch the output signal on the following graph:



5/5

a. Here is a resistor–capacitor circuit:

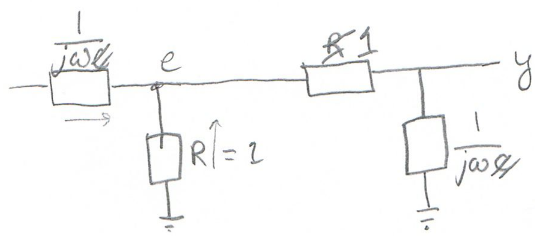


The input signal is $\cos t$ for all time (not just for $t \geq 0$). Choose the output signal from those given on the next page. *If you are unsure of the correct answer, you can also select a second answer, in which case you will receive the average of the scores for the two answers.*

1	2	3	4	5	6	7	8	9	10

Given cos (t) as v(t),

~~must be of 8/9 or 10 due to delay~~



0/5

e: ~~$v(t)\cdot j\omega$~~

$$(v(t)-e)\,j\omega + \frac{(0-e)}{} + (y-e) = 0$$

y: $$(0-y)\,j\omega + (e-y) = 0$$

$$v(t)\,j\omega - e(2+j\omega) + y = 0$$

$$e = \frac{y - v(t)\,j\omega}{2+j\omega}$$

$$-y\,j\omega + \frac{y - v(t)\,j\omega}{2+j\omega} - y =$$

$$y\left(-j\omega + \frac{1}{2+j\omega} - 1\right) = \frac{v(t)\,j\omega}{2+j\omega}$$

$$Y = \frac{\frac{v(t)\,j\omega}{2+j\omega}}{-j\omega + \frac{1}{2+j\omega} - 1}$$

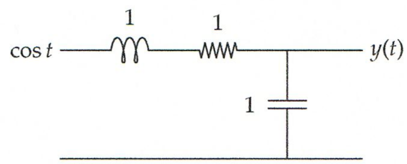
$$Y = v(t)\cdot\frac{j\omega}{(2+j\omega)\left(-j\omega + \frac{1}{2+j\omega} - 1\right)}$$

$$= v(t)\cdot\frac{j}{(2+j)(-j) + 1 - 2 - j}$$

$$= v(t)\cdot\frac{j}{-2j + 2 - 2 + j} = \frac{j}{-j} = \boxed{-1}$$

inverts the waveform at all frequencies

b. Here is an *LRC* circuit:



The input signal is $\cos t$ for all time (not just for $t \geq 0$). Choose the output signal from those given on the next page. *If you are unsure of the correct answer, you can also select a second answer, in which case you will receive the average of the scores for the two answers.*

1	2	3	4	5	6	7	8	9	10

$\frac{1}{2}$

$$Y(j\omega) = \frac{1/j\omega C}{j\omega L + 1 + 1/j\omega C}$$

$$= \frac{1/j}{j + 1 + 1/j}$$

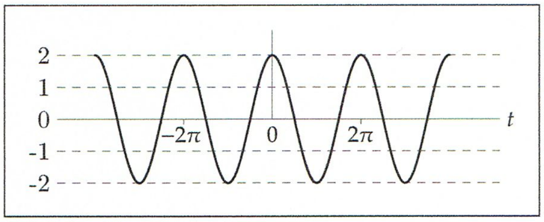
$$= \frac{1}{-1 + j + 1} = \frac{1}{j} = -j$$

$$|Y(j)| = 1$$

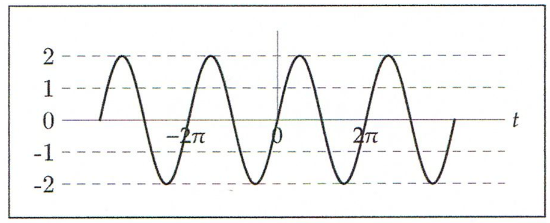
$$\measuredangle Y(j) = \pi/2$$

$\frac{2.5}{5}$

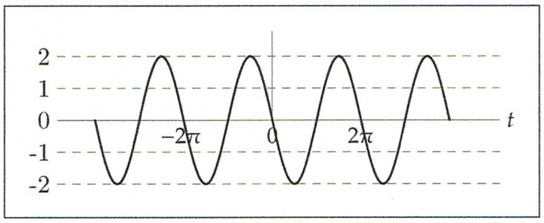
Here are the possible output signals for the preceding questions:



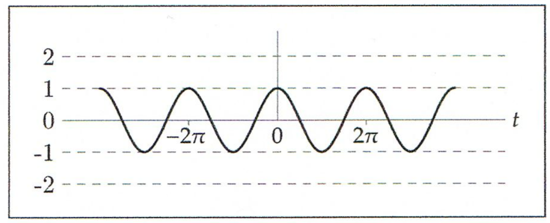
1



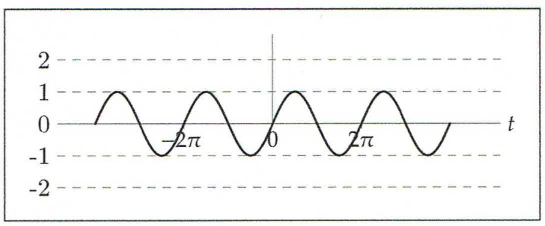
2



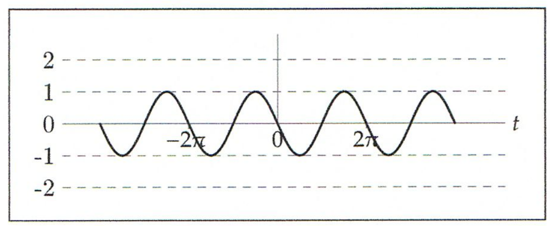
3



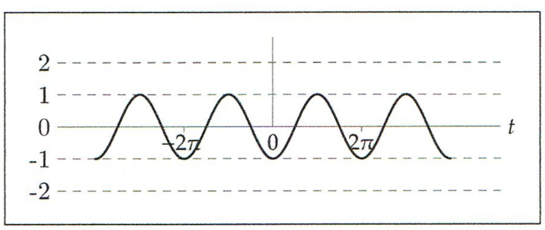
4



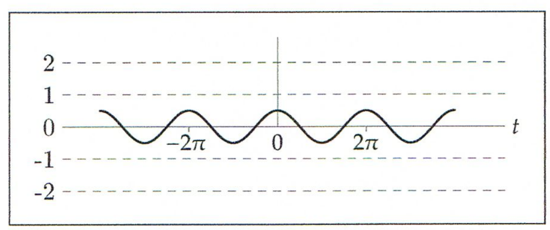
5



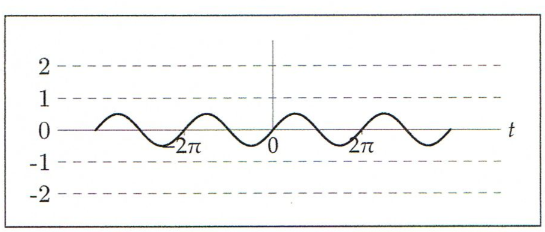
6



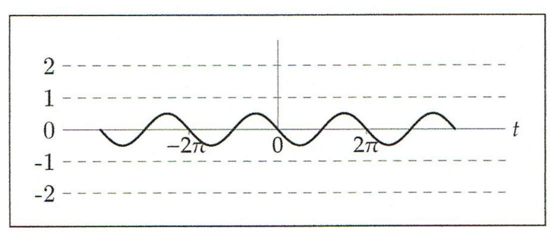
7



8



9



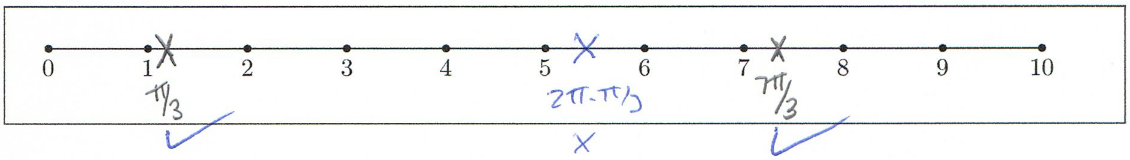
10

5. Mystery cosine *[5 points]*

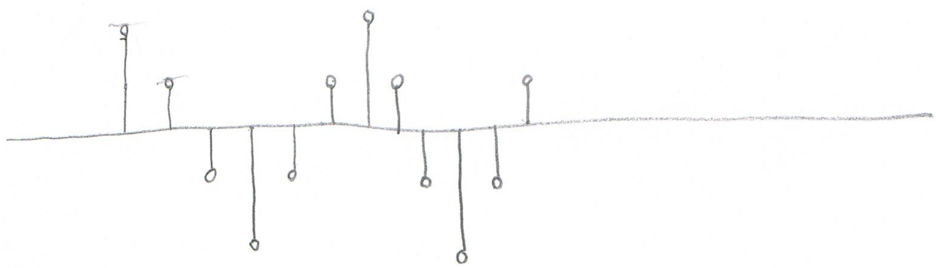
You sample $x(t) = \cos \omega t$ with sampling interval $T = 1$ and get this sequence of samples

$$\ldots, 1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \ldots$$

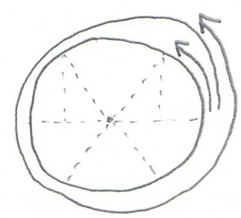
Using an **X**, mark all possible values of ω that lie in the thermometer range:



$2\pi - \pi/3$ missing



$x[n] = x(n)$



fundamental frequency:

4/5

$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \; 2\pi + \frac{\pi}{3}, \; 4\pi + \frac{\pi}{3}$

$\doteq 1.05 \quad \doteq 7.33$

6. Discrete-time poles and zeros *[10 points]*

A discrete-time system with a finite number of poles and zeros turns the causal signal

$$x[n] = \begin{cases} 3n+1 & \text{for } n \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

3 * unit ramp + 1

into the unit step

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Effect is of an derivative: first-difference,

$$y[n] = \frac{x[n] - x[n-1]}{3}$$

$$Y = \frac{1}{3}(X - \mathcal{R}X)$$

$$Y = \frac{1}{3}X(1-\mathcal{R})$$

$$H = \frac{Y}{X} = \tfrac{1}{3}(1-R)$$

one zero, no poles

a. What is the smallest number of zeros that the system could have? (If there are repeated zeros, count each.)

0	1	2	3	≥ 3

$H(z) = \frac{1}{3}(1 - R)$

If you are unsure of the correct answer, you can also select a second answer, in which case you will receive the average of the scores for the two answers.

b. What is the smallest number of poles that the system could have? (If there are repeated poles, count each.)

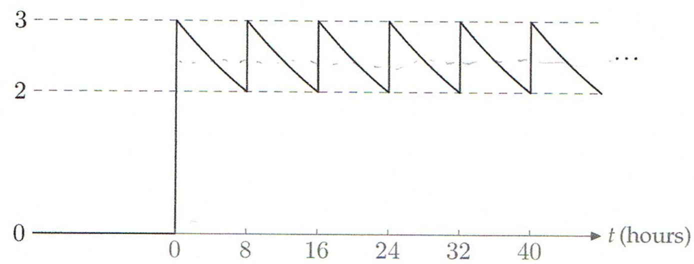
0	1	2	3	≥ 3

$H(z) = \frac{1}{3}(1 - z)$

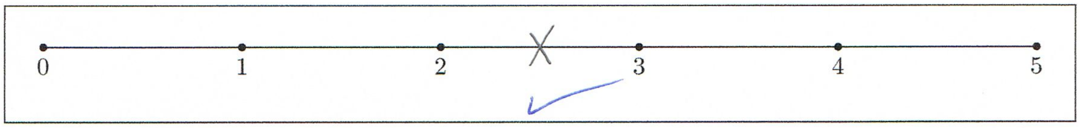
If you are unsure of the correct answer, you can also select a second answer, in which case you will receive the average of the scores for the two answers.

7. Taking medicine *[15 points]*

Starting at $t = 0$ a patient takes a medicine every 8 hours, eating a_0 pills at $t = 0$; a_1 pills at $t = 8$ hours; a_2 pills at $t = 16$ hours; and so on forever. The graph shows the concentration of medicine in the patient's blood as a function of time. This concentration is the output of a first-order system.

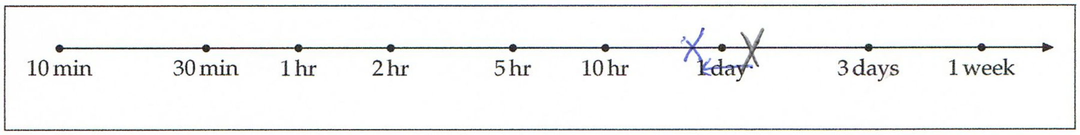


a. Mark the average concentration (in the region $t \geq 0$) as an **X** on the thermometer:



If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.

b. Mark the time constant of the first-order system as an **X** on the thermometer:



If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.

2.5/5 I'm giving part marks because no calculators were allowed, as a result my estimate was off slightly

$\tau =$

$Ae^{-t/\tau}$

@ $t=0$ $3e^0 = 3$, $A=3$

@ $t=8\text{hr}$ $3e^{-8/\tau} = 2$

$e^{-8/\tau} = \frac{2}{3}$

$e^{-8/\tau} = 2/3$

$-8/\tau = \ln 2 - \ln 3$

$\tau = \cancel{\frac{8}{\ln 2 - \ln 3}}$

$\frac{8}{\tau} \doteq \frac{1}{3}$

$\tau \doteq 24 \text{ hrs}$

$e^{-1} \doteq \frac{1}{2.7} \doteq 1/3$ $\tau > 8\text{hr}$

$e^{-0.5} \doteq \frac{1}{\sqrt{2.7}}$

$\frac{1}{\frac{3}{2}}$

$e^{-1/3}$ $\frac{1}{\sqrt[3]{2.7}}$

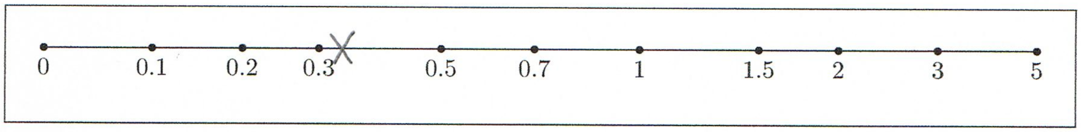
$\sqrt[x]{2.7} = 1.5$

$1.2 * 1.2 * 1.2 =$

1.44 $\quad$ 1.44 + .288 = 1.6... $\quad$ < 1.2

1.

c. Mark the ratio a_{2007}/a_0 as an **X** on the thermometer:



If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.

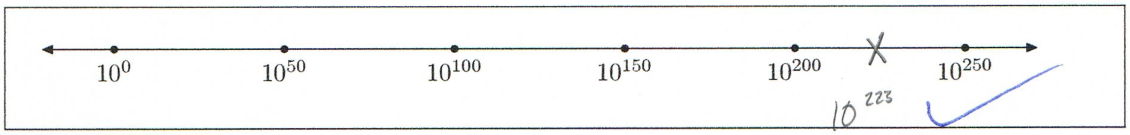
Since we're boosting with an impulse from $0 \to 3$ at a_0 and $2 \to 3$ at a_2

$= \frac{1}{3}$

8. Maximizing output samples *[5 points]*

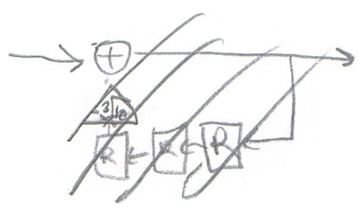
A discrete-time system has system functional $1/(1+\sqrt[3]{10}\mathcal{R}^3)$ with input signal X and output signal Y [the corresponding system function is $H(z) = z^3/(z^3+\sqrt[3]{10})$]. The input signal has two nonzero samples: $x[0]$ and $x[1]$. Choose $x[0]$ and $x[1]$ from the range $[-1, 1]$ in order to maximize $y[2007]$.

Mark the maximum value of $y[2007]$ as an **X** on the thermometer:



If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.

$$\frac{1}{1+\sqrt[3]{10}\,\mathcal{R}^3}$$



$$H(z) = \frac{z^3}{z^3+\sqrt[3]{10}}$$

poles @

$$y[n] + \sqrt[3]{10}\,y[n-3] = x[n]$$

$2007 \div 3 = 669$

$x = \langle 1, 1, 0 \ldots \rangle$

$y = \langle 1, 1, 0, \sqrt[3]{10}, \sqrt[3]{10}, 0, (\sqrt[3]{10})^2, (\sqrt[3]{10})^2, 0, \ldots$

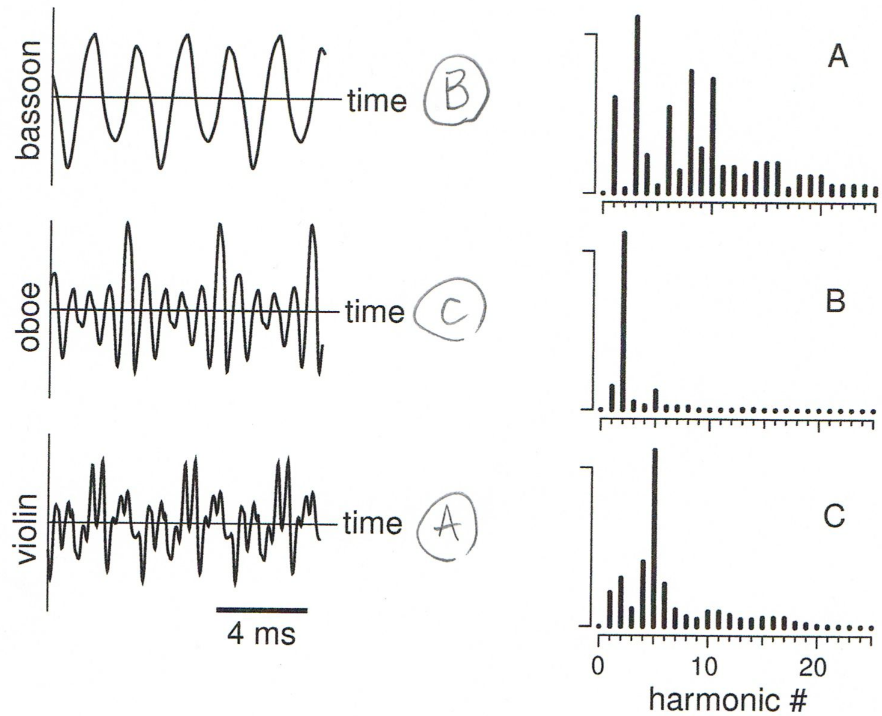
$y[2007] =$

$\frac{2007}{3} = 669$ multiplications of $\sqrt[3]{10}$

$$\boxed{= 10^{223}}$$

9. Musical timbre *[10 points]*

Time waveforms for three musical instruments each playing the same musical note are shown below (left panels, *each with same time scale*). Determine which Fourier series coefficients (right panels) correspond to the bassoon and the oboe.



a. Fourier series coefficients for the bassoon.

A B C

If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.

b. Fourier series coefficients for the oboe.

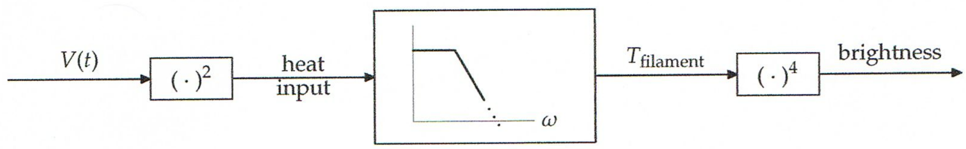
A B C

If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.

10. Lightbulb flicker *[10 points]*

Here is a model for variations in the brightness of a lightbulb.

Resistive heating pours heat into the lightbulb filament at a rate proportional to $V(t)^2$, where $V(t)$ is the line voltage oscillating at the line frequency $f = 60\,\text{Hz}$. The filament acts as a leaky tank with time constant $\tau \sim 30\,\text{ms}$ (in a typical 100 W bulb), turning the varying heat input into a varying temperature T_{filament}. The brightness of the lightbulb is proportional to T^4_{filament}.



T^4?

$\frac{1}{60} =$

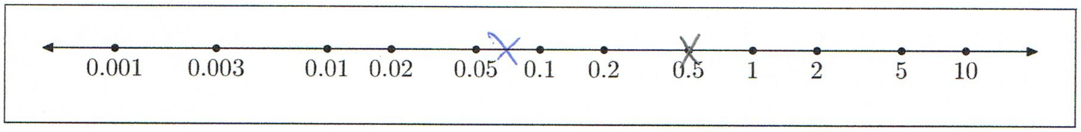
.01666
6 ⟌ 1.00
400

16 ms

a. Fractional ripple in the quantity γ is defined by

$$\text{fractional ripple in } \gamma \equiv \frac{\text{maximum value of } \gamma \; - \; \text{minimum value of } \gamma}{\text{average value of } \gamma}.$$

Mark the fractional ripple in T_{filament} as an **X** on the thermometer:

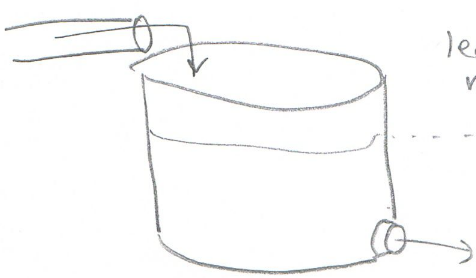


If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.

0.088

Fractional Ripple :

Max $T_{filament}$ =



leaky-tank model

$h = T_{Filament} = V(t)^2 * e^{-t/\tau}$

τ vs f

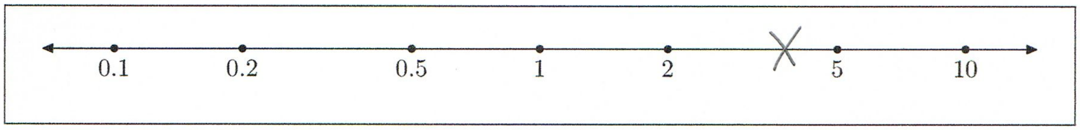
30 ms 16 ms, $T_{filament}$ drops about half of its max value

$\frac{.5}{.75} = \frac{2}{3} \sim 0.5$

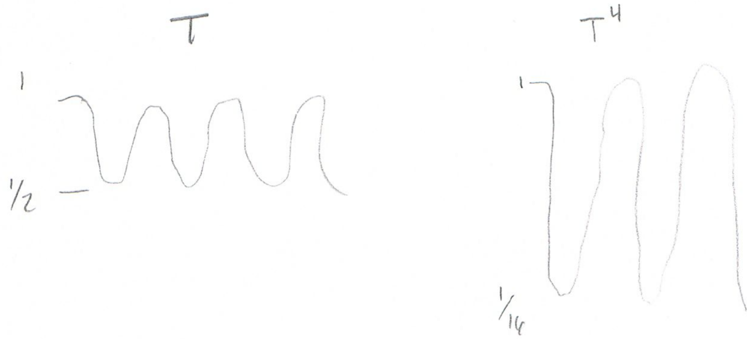
b. The brightness of a bulb is proportional to T^4_{filament}. What is the ratio

$$\frac{\text{fractional ripple in brightness}}{\text{fractional ripple in } T_{\text{filament}}}?$$

Mark the ratio as an **X** on the thermometer:



If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.



0.5

$$\frac{15/16}{8/16} = \frac{15}{8}$$

$$\frac{2}{0.5} \doteq 4$$ ✓

11. Half the unit step *[10 points]*

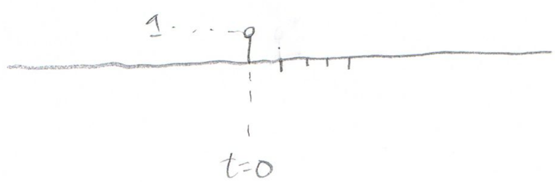
Consider a discrete-time signal $f[\cdot]$ that has no negative samples and satisfies

$$(f \star f)[n] = \text{the unit step } u[n] = 1, 1, 1, \ldots \text{ for } n \geq 0 \text{ (and zero otherwise).}$$

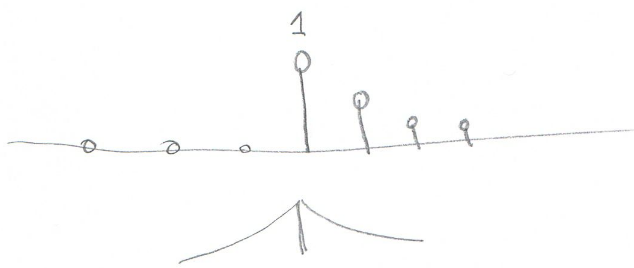
$f * f =$

$\cancel{F^2 = \text{unit step}}$

$\cancel{F = \sqrt{\text{unit step}}}$



$1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{1\ldots}$



$(f*f)[0] = 1 * 1 = 1$

$\text{"} \quad [1] = 1 \cdot 0.5 + 1 \cdot 0.5$

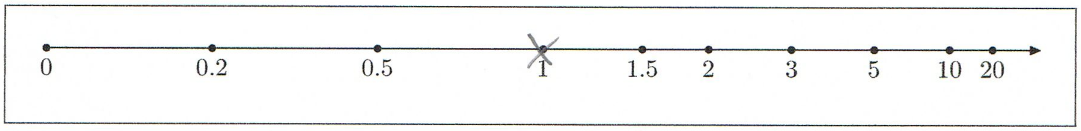
$[2] = 1 \cdot (x) + (0.5)(0.5) + 1 \cdot x = 1$

$0.75 = 2x \qquad x = \frac{3}{8}$

$[3] =$

$2(0.5)3/8 + 2x = 1 - 3/8 = 5/8 \qquad 5/16$

a. Mark $f[0]$ as an **X** on the thermometer:



$\frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64}$

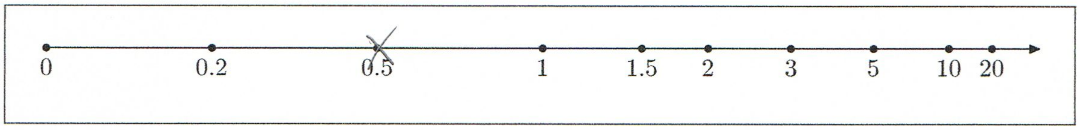
$\boxed{4} = 2x + \frac{20}{64} + \frac{9}{64} = 1$

$2x + \frac{29}{64} = 1$

$2x = \frac{35}{64} = \frac{35}{128}$

$64 - 29 =$

b. Mark $f[1]$ as an **X** on the thermometer:



If you are unsure of the correct value, you can also mark a second value, in which case you will receive the average of the scores for the two values.