

6.041/6.431 Spring 2006 Final Exam

Monday, May 22, 1:30-4:30PM

DO NOT TURN THIS PAGE OVER UNTIL
YOU ARE TOLD TO DO SO

Name:

SCOTT YOUNG

Recitation Instructor:

TA:

Question	Score	Out of
0		
1	38.5	42
2	15	30
3	18	28
Your Grade	71.5%	100

71.5%

- You have 3 hours to complete the exam.
- At the end of the exam period you will turn in this exam packet, and 2 blue books. Question 1 will be answered in the exam packet, while question 2 and 3 will be answered in their own respective blue books.
- Question 1 is multiple choice, no partial credit is given.
- For questions 2 and 3, you should concisely indicate your reasoning and show all relevant work. Grades will be based on our judgment of your level of understanding as reflected by what you have written. Writing the correct answer without clearly showing your understanding of how to arrive at the answer will receive no credit.
- Answers in the form of an arithmetic expression (sums, products, ratios, factorials) should be fully reduced to receive full credit.
- This is a closed-book exam except for three double-sided 8.5" by 11" formula sheets.
- Calculators are allowed.
- Be neat! If we can't read it, we can't grade it.

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Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Problem 0:

Write your name, your recitation instructor's name, and TA's name on the cover of the exam booklet and your 2 blue books. Include the question number on the cover of the blue book.

Problem 1: (42 points)

Multiple Choice Questions: There is only one correct answer for each question listed below, please clearly indicate the correct answer. There will be **no partial credit** given for multiple choice questions, thus any explanations will not be graded. Each question is equally weighted at 3.5 points each.

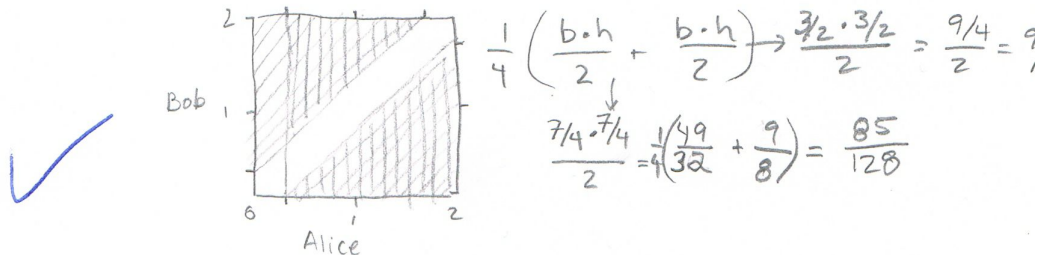
- (I) Alice and Bob each choose a number independently and uniformly at random from the interval $[0, 2]$. Consider the following events:

A : The absolute difference between the two numbers is greater than $1/4$.

B : Alice's number is greater than $1/4$.

Find $P(A \cap B)$.

- (a) $\frac{49}{128}$
(b) $\frac{75}{128}$
(c) $\frac{85}{32}$
(d) $\frac{85}{128}$



- (II) There are m red balls and n white balls in an urn. We draw two balls simultaneously and at random. What is the probability that the balls are of different color?

- (a) $\frac{mn}{(m+n)(m+n-1)}$
(b) $\frac{2mn}{(m+n)(m+n-1)}$
(c) $\frac{1}{mn}$
(d) $\frac{2}{mn}$

Handwritten calculation:

$$\frac{2m}{(m+n)} \cdot \frac{n}{(m+n-1)}$$

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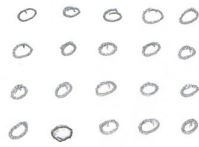
- (III) 20 black pebbles are arranged in 4 rows of 5 pebbles each. We choose 4 of these pebbles at random and color them red. What is the probability that all the red pebbles lie in different rows?

(a) $\frac{5^4 \cdot 5! \cdot 15!}{20!}$

(b) $\frac{4^5 \cdot 5! \cdot 15!}{20!}$

(c) $\frac{5^4 \cdot 4! \cdot 16!}{20!}$

(d) $\frac{4^5 \cdot 4! \cdot 16!}{20!}$



$\frac{4 \cdot 4 \cdot 4 \cdot 4}{20!}$
 $\frac{4!}{4! \cdot 16!}$

$5 \cdot 5 \cdot 5 \cdot 5$

$\binom{20}{4}$ ways of picking 4 stones

$4 \binom{5}{1}$ ways of picking a stone in each row

- (IV) We have two light bulbs, A and B. Bulb A has an exponentially distributed lifetime with mean lifetime 4 days. Bulb B has an exponentially distributed lifetime with mean lifetime 6 days. We select one of the two bulbs at random; each bulb is equally likely to be chosen. Given that the bulb we selected is still working after 12 hours, what is the probability that we selected bulb A?

(a) $\frac{1}{1 + e^{\frac{1}{24}}}$

(b) $\frac{1}{1 + e^{-1}}$

(c) $\frac{1}{1 + e^1}$

(d) $\frac{1}{1 + e^{-\frac{1}{24}}}$

$X =$ still working after 12 hours

$A =$ chose bulb A

$$P(A|X) = \frac{P(A)P(X|A)}{P(X)} = \frac{0.5(1 - e^{-2})}{(0.5)(1 - e^{-2} + 1 - e^{-3})}$$

≈ 0.47

$\int_0^{0.5} 4e^{-4x} dx = -e^{-4x} \Big|_0^{0.5} = 1 - e^{-2}$

- (V) A test for some rare disease is assumed to be correct 95% of the time: if a person has the disease, the test results are positive with probability 0.95, and if the person does not have the disease, the test results are negative with probability 0.95. A random person drawn from a certain population has probability 0.001 of having the disease. Given that the person just tested positive, what is the probability of having the disease?

(a) 0.00095

(b) 0.001

(c) 0.0187

(d) 0.128

$$P(D|+) = \frac{P(D)P(+|D)}{P(+)}$$

$$= \frac{0.001 \cdot 0.95}{(0.001 \cdot 0.95) + (0.999 \cdot 0.05)}$$

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(VI) Let X be a random variable with

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = 2X - 4$. Find $f_Y(0)$.

- (a) $1 - e^{-4}$
- (b) $2e^{-4}$
- (c) $2e^{-8}$
- ☒ (d) e^{-4}

$e^{-y-4} = e^{-y} e^{-4}$
 $f_Y(0) = e^{-4}$

$f_Y = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$
 $= \frac{1}{2} f_X\left(\frac{y+4}{2}\right) = \frac{1}{2} \cdot 2e^{-2\left(\frac{y+4}{2}\right)} = e^{-y-4}$

(VII) X and Y are independent random variables, with

$$f_X(x) = \begin{cases} 3e^{-3x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad p_Y(y) = \begin{cases} 1/2 & \text{if } y = 1 \\ 1/2 & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = X + Y$. What is the transform of Z , $M_Z(s)$, evaluated at $s = 1$?

- (a) $3(1+e)/2$
- ☒ (b) $3(1+e)/4$
- (c) $3e(1+e)/2$
- (d) $3e(1+e)/4$

$M_Z = M_X M_Y$
 $M_X = \frac{3}{3-s}$
 $M_Z = \frac{3}{3-s} \left(\frac{1}{2} + \frac{1}{2} e^s \right)$

$M_Z|_{s=1} = \frac{3}{2} \left(\frac{1}{2} + \frac{1}{2} e \right) = \frac{3}{4} (1+e)$
 $M_Y = e^{s(0)} \left(\frac{1}{2} \right) + e^{s(1)} \left(\frac{1}{2} \right) = \frac{1}{2} + \frac{1}{2} e^s$

(VIII) A number p is drawn from the interval $[0, 1]$ according to the uniform distribution, and then a sequence of independent Bernoulli trials is performed, each with success probability p . What is the variance of the number of successes in k trials? Note k is a deterministic number.

- (a) $\frac{k}{2}$
- (b) $\frac{k(1+k)}{4}$
- ☒ (c) $\frac{k(2+k)}{12}$
- (d) $\frac{k}{4}$

variance = $np(1-p) = kp(1-p)$
 $\text{var}(p) = \frac{1}{12}$

$E[\text{var}(X)] = \int_0^1 kp(1-p) dp = \int_0^1 kp - kp^2 dp$
 $= \left[\frac{kp^2}{2} - \frac{kp^3}{3} \right]_0^1 = \frac{k}{2} - \frac{k}{3} = \frac{k}{6}$

$\text{var}(X) = E[\text{var}(X|p)] + \text{var}(E[X|p])$
 $= E[kp(1-p)] + \text{var}(kp)$
 $= \frac{k}{4} + \frac{k^2}{12}$

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- (IX) A police radar always over-estimates the speed of incoming cars by an amount that is uniformly distributed between 0 and 5 mph. Assume that car speeds are uniformly distributed between 60 and 75 mph and are independent of the radar over-estimate. If the radar measures a speed of 76 mph, what is the least squares estimate of the actual car speed?

- (a) 72.5 mph
(b) 73.0 mph
(c) 73.5 mph
(d) 72.9 mph

estimate, add 2.5

$$76 - 2.5 = 73.5$$

X

- (X) You are visiting the Killian rainforest, when your insect repellent runs out. This forest is infested with lethal mosquitoes, whose second bite will kill any human instantaneously. Assume mosquito's land on your back and deliver a vicious bite according to a Bernoulli Process. Also assume the expected time till the first bite is 10 seconds. If you arrive at the forest at $t = 0$, what is the probability that you die at exactly $t = 10$ seconds?

(a) $\frac{9^9}{10^{10}}$

(b) $\frac{9^8}{10^{10}}$

(c) $\binom{9}{1} \frac{1}{10} \left(\frac{9}{10}\right)^8$

(d) $\frac{1}{2}$

$$p = \frac{1}{10}$$

$$P_T = (1-p)^{t-1} (p)$$

$$P_T(10) = \left(\frac{9}{10}\right)^9 \left(\frac{1}{10}\right)$$

- (XI) In order to estimate p , the fraction of people who will vote for George Bush in the next election, you conduct a poll of n people drawn randomly and independently from the population. Your estimator M_n is obtained by dividing S_n , the number of people who vote for Bush in your sample, by n , i.e., $M_n = S_n/n$. Find the smallest value of n , the number of people you must poll, for which the Chebyshev inequality yields a guarantee that

$$P(|M_n - p| \geq 0.01) \leq 0.01.$$

$$\sigma^2 = np(1-p)$$

(Use the fact that $p(1-p) \leq 1/4$ for any $p \in [0, 1]$.)

- (a) 500,000
(b) 250,000
(c) 25,000
(d) 50,000

✓

$$P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$$

$$P(|M_n - p| \geq 0.01) \leq \frac{\sigma^2}{(0.01)^2} \leq \frac{n \left(\frac{1}{4}\right)}{n(0.01)^2} = 0.01$$

$$\frac{\left(\frac{1}{4}\right)}{n(0.01)^2} \leq 0.01 \quad \frac{1}{4n} = (0.01)^2$$

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- (XII) On each day of the year, it rains with probability 0.1, independent of every other day. Use the Central Limit Theorem (CLT) (with no half step) to approximate the probability that, out of the 365 days in the year, it will rain on at least 100 days. In the answers below, Φ denotes the standard normal CDF.

(a) $1 - \Phi\left(\frac{640}{3\sqrt{365}}\right)$

(b) $\Phi\left(\frac{640}{3\sqrt{365}}\right)$

(c) $\Phi\left(\frac{635}{3\sqrt{365}}\right)$

(d) $1 - \Phi\left(\frac{635}{3\sqrt{365}}\right)$

$$Z = \frac{(c - n\mu)}{\sigma\sqrt{n}} = \frac{(100 - 36.5)}{0.09\sqrt{365}}$$

$$200 - 73$$



$$\sigma = 0.09$$

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Problem 2: (30 points)

Please write all work for Problem 2 in your **first blue book**. No work recorded below will be graded. Each question is equally weighted at 6 points each.

You take a safari trip to the Porobilati game reserve. A highlight of the game reserve is the Poseni river where one can watch deer and elephants coming to drink water. Deer come to the river according to a Poisson process with arrival rate $\lambda_d = 8$ per hour; elephants come to the river according to an independent Poisson process with arrival rate $\lambda_e = 2$ per hour. On the first day of your safari, you reach the Poseni river early in the morning hoping to see some elephants. Assume that deer and elephants are the only kinds of animals that visit this river.

- (a) Let N be the number of animals you see during the first 3 hours. Find $\mathbf{E}[N]$ and $\text{var}(N)$.
-  (b) What's the probability of seeing your 3rd elephant before your 9th deer?
- (c) At the end of 3 hours, you have seen 24 deer but no elephants. What is the probability that you will see an elephant in the next hour?
- (d) It is now 6 hours since you started watching. You have seen 53 deer but there is still no sight of an elephant. How many more deer do you expect to see before you see your first elephant?
-  (e) Unfortunately, you forgot your camera in your lodge the first day. You come back with your camera the next day and plan to stay at the river until you've clicked a picture of both a deer and an elephant. Assume you're always able to click a picture as soon as an animal arrives. How long do you expect to stay?
- (f) Your friend, a wildlife photographer, is fascinated by the stories of your trip, and decides to take the safari herself. All excited, she arrives at the Poseni river, and is prepared to stay as long as required in order to get lots of beautiful pictures. Unfortunately, she dropped her camera in a bush on the way, as a result of which the camera is not functioning properly; each time she clicks, there is a probability 0.1 that the camera will produce a faulty picture, independently of everything else. Assume she clicks one picture each time an elephant or deer arrives at the river, as soon as it arrives. Let X be the time (in hours) until she gets her third successful picture of an elephant. Find the PDF of X .

PROBLEM 2:

a) $N = X + Y$
 $\uparrow \quad \uparrow$
 $\# \text{ of elephants} \quad \# \text{ of deer}$

$$M_N = M_X M_Y$$

$$= e^{\lambda_e(z)(e^z-1)} e^{\lambda_d(z)(e^z-1)}$$

$$= e^{z(\lambda_e(e^z-1) + \lambda_d(e^z-1))}$$

$$= e^{z(\lambda_e + \lambda_d)(e^z-1)}$$

$N = \text{Poisson w/ param } 3(\lambda_e + \lambda_d)$ ✓

$$E[N] = 3(\lambda_e + \lambda_d) = 30$$

$$\text{Var}(N) = 3(\lambda_e + \lambda_d) = 30$$

c) $\int_0^1 \lambda_e e^{-\lambda_e x} dx$ Time until first arrival is exponential w/ λ_e

$$= -e^{-\lambda_e x} \Big|_0^1$$

$$= 1 - e^{-\lambda_e} = 1 - e^{-2}$$
 ✓

e) $f_{Y_t} = \text{PDF of time until deer} = \lambda_d e^{-\lambda_d t}$

$$f_{X_t} = \lambda_e e^{-\lambda_e t}$$

$$E[T] = \int_0^\infty \int_0^\infty \lambda_e e^{-\lambda_e t} dt$$



(b) $P_{Y_9} = \frac{\lambda_d^9 e^{-\lambda_d}}{9!} = \frac{8^9 e^{-8}}{9!}$

$$P_{X_3} = \frac{\lambda_e^3 e^{-\lambda_e}}{3!} = \frac{27 e^{-3}}{6}$$

$$P(X_3 < Y_9) =$$

$$P(Y_9 - X_3 > 0), \quad Z = Y_9 - X_3$$

$$\text{CDF}(Z) = \int$$

(d) $E[\text{time until elephant}] = 1/2 \text{ hr}$

$$E[\# \text{ of deer in } 1/2 \text{ hr}] = 4 \text{ deer}$$
 ✓

(f) Time until 3rd elephant = T_{3E}

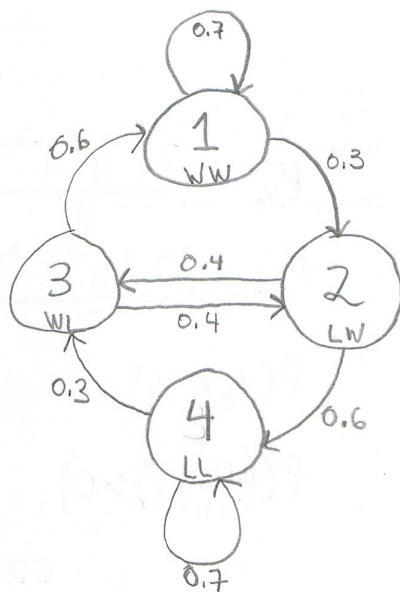
$$E[T_{3E}] = \frac{3}{\lambda_e}$$

each elephant has a 0.9% chance of being photographed, so we expect to wait for $\frac{3}{0.009} = 333$ "elephants" before getting 3 pictures

$$\frac{10/3}{2} = 5/3 \text{ hours}$$

PROBLEM 3:

a)



WW = 1 = won last two games
LW = 2 = lost last game, won before that
WL = 3 = won last game, lost before that
LL = 4 = lost last two games

b) 0.6 → corresponds to 2 → 4 transition

c)

$$f_x(k) = (0.7)^{k-1} (0.3)$$

↑ Geometric w/ $p = P(\text{Loss w/ WW})$

d)

$$\begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.7 & 0 & 0.6 & 0 \\ 0.3 & 0 & 0.4 & 0 \\ 0 & 0.4 & 0 & 0.3 \\ 0 & 0.6 & 0 & 0.7 \end{bmatrix} \end{matrix}$$

There is no steady state because the chain is periodic.

X

e) $P(X) = 0.5$, assuming WW, WL, LW and LL are equally likely for games #998 and #999. This can be easily confirmed given the chain is symmetric

f) This is the equivalent of calculating the first passage time of state #4

$$\boxed{\frac{1}{2}}$$

$$t_1 = 0$$

$$t_2 = 1 + 0.3t_1 + 0.4t_3$$

$$t_3 = 1 + 0.3t_4 + 0.4t_2$$

$$t_4 = 1 + 0.7t_4 + 0.6t_2$$

g) This is equivalent to the mean recurrence time for state #4, plus the first-passage time from (f)

$$S_4^* = 1 + 0.7t_4 + 0.3t_3$$

$$E[N] = S_4^* + t_4$$

X

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Problem 3: (28 points)

Please write all work for Problem 3 in your **second blue book**. No work recorded below will be graded. Each question is equally weighted at 4 points each.

The MIT football team's performance in any given game is correlated to its morale. If the team has won the past two games, then it has a 0.7 probability of winning the next game. If it lost the last game but won the one before that, it has a 0.4 probability of winning. If it won its last game but lost the one before that it has a 0.6 probability of winning. Finally if it lost the last two games it has only a 0.3 probability of winning the next game. No game can end up in a draw. Consider a starting time when the team has won its preceding two games.

- (a) Graph the minimum state Markov chain model for the MIT football team's performance. Be sure to label all transition probabilities, and clearly indicate what state of the world each Markov state represents. You may find it convenient for the below questions to label each state with an integer.
- (b) Find the probability that the first future loss will be followed by another loss.
- (c) Let X be the number of games played up to, but not including the first loss. Find the PMF of X .
- (d) Either evaluate the steady-state probabilities or explain why they do not exist.
- (e) Find a good approximation to the probability that the team will win its 1000th game, given that the outcomes of games 1000 and 1001 are the same. Clearly state any assumptions you use in the approximation.
- (f) Let T be the number of games up to and including the team's 2nd consecutive loss (i.e. 2 losses in a row). Write a system of equations that can be used to find $\mathbf{E}[T]$.
- (g) The coach decides that he will withdraw the team from the competition immediately after the team's 3rd consecutive loss (i.e. three losses in a row). Let N be the number of games that the team will play in the competition? Write a system of equations that can be used to find $\mathbf{E}[N]$.

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c \quad \left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c$$

6.041 Formula Sheet

COUNTING

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \leftarrow \text{choosing}$$

$$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1! k_2! k_3!} \leftarrow \text{partitioning}$$

PROBABILITY LAWS

$$\text{If } A \subseteq B, P(A) \leq P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) \leq P(A) + P(B)$$

$$P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A_1 \cup A_2 | B) = P(A_1 | B) + P(A_2 | B)$$

\hookrightarrow for A_1, A_2 disjoint

COMMON DISTRIBUTIONS

AND THEIR PROPERTIES:

~~BERNOULLI~~ (sum of coin flips)

$$\text{BINOMIAL: } f_x = \binom{n}{k} p^k (1-p)^{n-k} \quad E[X] = np$$

$$\text{var}(X) = np(1-p)$$

$$M(s) = (1-p+pe^s)^n$$

POISSON: (arrivals in a lineup) \rightarrow good approx for binomial when n is large & p is small

$$f_x = e^{-\lambda} \frac{\lambda^k}{k!} \quad E[X] = \lambda \quad \text{var}(X) = \lambda \quad M(s) = e^{\lambda(e^s - 1)}$$

EXPONENTIAL:

$$f_x = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases} \quad E[X] = \frac{1}{\lambda} \quad \text{var}(X) = \frac{1}{\lambda^2} \quad M(s) = \frac{\lambda}{\lambda - s}$$

NORMAL:

$$f_x = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad E[X] = \mu \quad E[Y] = a\mu + b \quad \Phi\left(\frac{x-\mu}{\sigma}\right) \quad \text{var}(X) = \sigma^2 \quad \text{var}(Y) = a^2\sigma^2 \quad M(s) = e^{(\sigma^2 s^2/2) + \mu s}$$

GEOMETRIC: (time until first heads)

$$f_x = (1-p)^{k-1} p \quad E[X] = \frac{1}{p} \quad \text{var}(X) = \frac{1-p}{p^2} \quad M(s) = \frac{pe^s}{1-(1-p)e^s}$$

UNIFORM:

$$f_x = \begin{cases} \frac{1}{b-a+1}, & \text{if } a \leq k \leq b \\ 0, & \text{else} \end{cases} \quad E[X] = \frac{a+b}{2} \quad \text{var}(X) = \frac{(b-a)(b-a+2)}{12} \quad \text{continuous} \quad \frac{(b-a)^2}{12} \quad M_{\text{cont}}(s) = \frac{1}{b-a} \cdot \frac{e^{sb} - e^{sa}}{s} \quad M_{\text{disc}}(s) = \frac{e^{as}}{b-a+1} \cdot \frac{e^{(b-a+1)s} - 1}{e^s - 1}$$

BERNOULLI: (one coin flip)

$$f_x = \begin{cases} p, & x=1 \\ 1-p, & x=0 \end{cases} \quad E[X] = p \quad \text{var}(X) = p(1-p) \quad M(s) = 1-p+pe^s$$

RANDOM VARIABLES, EXPECTATION AND VARIANCE

$$E[X] = \sum_x x p_x(x) \quad \text{var}(X) = E[X^2] - (E[X])^2 \quad E[X|A] = \sum x p_{X|A}(x)$$

$$\text{var}(X) = E[(X - E[X])^2] = \sum (x - E[X])^2 p_x(x)$$

$$E[E[X|Y]] = E[X]$$

$$Y = aX + b$$

$$E[X] = \int x f(x) dx$$

$$\text{var}(X) = E[\text{var}(X|Y)] + \text{var}(E[X|Y])$$

$$E[Y] = aE[X] + b$$

$$\text{var}(Y) = a^2 \text{var}(X)$$

$$f_Y = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$(N = \# X_i)$$

$$E[Y] = n\mu$$

$$\text{var}(Y) = E[N] \text{var}(X) + (E[X])^2 \text{var}(N)$$

The transform of M_Y is found by replacing each e^s with $M_{X_i}(s)$

Transforms

$$M(s) = \sum e^{sx} p_x(x) = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx$$

$$\frac{d}{ds} M(s) \Big|_0 = E[X] \quad \frac{d^n}{ds^n} M(s) \Big|_0 = E[X^n]$$

$$M_W = M_X M_Y \\ W = X + Y$$

Linear Transform:

$$M_Y(s) = e^{sb} M_X(sa)$$

if X is nonnegative:

$$\lim_{s \rightarrow -\infty} M_X(s) = P(X=0)$$

LIMIT THEOREMS:

$$\text{Markov} \Rightarrow P(X \geq a) \leq \frac{E[X]}{a} \quad (X \text{ is nonnegative})$$

$$\text{Chebyshev} \Rightarrow P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$$

CORRELATION, COVARIANCE & REGRESSION

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X) \text{var}(Y)}}$$

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$

$E[(X-c)^2]$ is minimized when $E[X] = c$

Variance \leq square error, for all c .

$$\text{var}(X) = \text{var}(\hat{X}) + \text{var}(\tilde{X})$$

↑ ↑
estimate error

Linear LSR:

$$E[X] + \rho \frac{\sigma_X}{\sigma_Y} (Y - E[Y])$$

w/ error of

$$(1 - \rho^2) \sigma_X^2$$

Normal Approx to Binomial

$$\text{mean} \rightarrow np \\ z = \frac{(c - np)}{\sigma \sqrt{n}}$$

$$P(S_n \leq c) \approx \Phi(z)$$

WEAK LAW

→ There's a low chance of being away from the mean

STRONG LAW

→ finite deviations from the mean can only occur a finite number of times

MARKOV CHAINS

n -step transition probabilities =

$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1) p_{kj} \quad \left\{ \begin{array}{l} \text{alternatively it is the transition matrix } A \text{ raised to the power of } n \end{array} \right.$$

with $r_{ij}(1) = p_{ij}$

steady-state & stationary distribution:

$$\lim_{n \rightarrow \infty} r_{ij}(n) = \pi_j \quad \leftarrow \text{This is the steady-state probability}$$

$$\pi_j = \sum_{k=1}^m \pi_k p_{kj} \quad \left\{ \begin{array}{l} \text{or, this system of equations} \end{array} \right.$$

$$1 = \sum_{k=1}^m \pi_k \quad \left\{ \begin{array}{l} \text{balance equations} \\ \text{normalization equation} \end{array} \right.$$

$$\lim_{n \rightarrow \infty} \frac{f_{jk}(n)}{n} = \pi_j p_{jk}$$

expected frequency of a transition

ABSORPTION

$$a_s = 1 \\ a_i = 0, \text{ if } i \text{ is absorbing}$$

$$a_i = \sum_{j=1}^m p_{ij} a_j, \text{ for } i \text{ not absorbing}$$

$$E[T] \text{ if } \mu_i = 0 \text{ for } i \text{ absorbing}$$

$$\mu_i = 1 + \sum_{j=1}^m p_{ij} \mu_j$$

In a birth-death process:

$$\pi_i = \pi_0 \frac{b_0 b_1 b_2 \dots b_{i-1}}{d_1 d_2 d_3 \dots d_i}$$

First Passage time:

$$t_s = 0, \quad t_i = 1 + \sum_{j=1}^m p_{ij} t_j$$

Mean Recurrence time: (Repeat)

$$t_s^* = 1 + \sum_{j=1}^m p_{sj} t_j$$

PROCESSES

Bernoulli Process:

$$P_s(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \left\{ \begin{array}{l} \# \text{ of successes} \end{array} \right.$$

$$E[s] = np$$

$$\text{var}(s) = np(1-p)$$

$$P_T(t) = (1-p)^{t-1} p \quad \left\{ \begin{array}{l} \text{time until 1st success} \end{array} \right.$$

$$E[T] = \frac{1}{p} \quad \text{var}(T) = \frac{1-p}{p^2}$$

$$P_Y(k) = \binom{t-1}{k-1} p^k (1-p)^{t-k} \quad \left\{ \begin{array}{l} k^{\text{th}} \text{ arrival time} \end{array} \right.$$

$$E[Y_k] = \frac{k}{p} \quad \leftarrow \text{Pascal PMF}$$

$$\text{var}(Y_k) = \frac{k(1-p)}{p^2}$$

Poisson Process:

of arrivals is Poisson with parameter λ .

time until first arrival is exponential with λ

k^{th} arrival =

$$E[Y_k] = \frac{k}{\lambda} \quad \text{var}(Y_k) = \frac{k}{\lambda^2}$$

$$P_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!} \quad \leftarrow \text{Erlang PDF}$$