NOTES COMPRESSION & CRIB SHEET

1) Extensionality: A = B if same elems
2) Pairing:
3) Union
4) Infinity
5) Power set
6) Replacement
7) Foundation
8) Choice

Symbols:
::= assignment
$\wedge \to$ &&
$\vee \to$ ||
$\neg \to$!
$\oplus \to$ XOR

ORDER OF LOGIC MATTERS:

$\exists d \in D\ \forall a \in A.\ H(a,d)$
"There is a single dream all Americans share."

vs

$\forall a \in A\ \exists d \in D.\ H(a,d)$
"For every American, there is a dream"

validity = true for all values of P(i)

SET NOTATION
$\in$ → element of
$\forall$ → All
$\exists$ → There exists
$\cup$ → union of sets
$\cap$ → intersection of sets
$\bar{S}$ → complement
$\mathcal{P}$ → Power set (set of all subsets)
{ } → set
( ) → sequence
λ → empty set/sequence
$\subset$ → subset of

PROOFS:
→ PROPERTIES TO CHECK FOR:
* EVENNESS
* PRIME NUMBERS

2) look for smallest C
3) Show through contradiction that C is not smallest
4) conclude C is empty

Binary Relations
Total → all A have B
Surjective → mapped at least once (every B has at least one ...)
Injective → mapped at most once
Bijective → paired

if A surj B then $|A| \geq |B|$

if A inj B then $|A| \leq |B|$

MAKING CHANGE
1) PROVE BASE CASES
2) PROVE P(n + small coin) given P(n)

INDUCTION
1) state Base Case
2) Assume P(n)
3) Prove P(n+1) follows from P(n)

STRONG INDUCTION
4) Assume P(m) m≤n
5) Prove P(n+1), given P(0)...P(n) are true

PARTIAL ORDERS
transitivity
if aRb & bRc → aRc
= is transitive
≠ is not transitive
symmetry:
if a→b then b→a
strict partial order (<) is transitive & assymmetric
reflexive:
if a→a ∀a
assymmetry: < >
antisymmetry: ≤ ≥

PRODUCT ORDER
$(a,b)(R \times S)(c,d) \longleftrightarrow [aRc \ \&\ bSd]$
preserves these, but not totality
chain has comparable elements
antichains have incomparable elems

Dilworth's Lemma:
If largest chain is |t| then t antichains can be made
↳ for all t>0, every poset must have a chain ≥ t or an antichain of at least n/t

DIGRAPHS
DAG, D⁺ is a strict poset

STATE MACHINES
Reach states → check if a preserved invariant is violated (unreachable)
partial correctness → final results are always correct
+
termination → the SM eventually stops, always
↳ look for a derived variable which is strictly decreasing

Euclid's GCD:
$(x,y) \to (y, \text{rem}(x,y))$

Stable Marriage
↳

Graphs
adjacent vertices
incident edges

Handshake Lemma:
$\sum \deg(v) = 2e$
simple path means no repeat visits
simple cycle is a non-repeating loop

k-connected requires k edges to be deleted before disconnecting

Every Graph has at least v − e connected components.

Bipartite Graph
↳ 2-colorable

COLORING
G with max degree k is k+1 colorable

TREES
connected graph w/o cycles

RECURSIVE DATA TYPES
Base Case
Constructor Case

Aexp:
Substitution Model:
↳ substitute & evaluate
Environment Model:
↳ evaluate & substitute

Hall's theorem:
Matching of girls and boys iff every subset of girls likes at least as large a set of boys

continuous faces (don't forget outside face!)

$v - e + f = 2$ every planar graph has
$e \leq 3v - 6$ max(deg) ≤ 5
$3f \leq 2e$ (5-colorability)

NETWORKS
latency = longest path

$\frac{1}{m} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2}$ } polyhedra
m: # faces at a corner
n: sides per face
e: edges

GAMES
↳ use structural induction to prove termination

NUMBER THEORY
1) $a|b \to a|bc\ \forall c$
2) $a|b\ \&\ b|c \to a|c$
3) $a|b\ \&\ a|c \to a|sb + tc\ \forall s,t$
4) $c \neq 0\quad a|b \leftrightarrow ca|cb$

MODULAR ARITHMATIC
main difference is mult. don't cancel (except for w/ primes!)
multiplicative inverse
$sp + tk = 1$
Fermat's Little Theorem:
$k^{-1} = \text{rem}(t, p)$
$k^{p-1} \equiv 1 \pmod p$
$\phi(n)$ = # of integers relatively prime to n
$\phi(ab) = \phi(a)\phi(b)$
$\phi(p^k) = p^k - p^{k-1}$
$k^{\phi(n)} \equiv 1 \pmod n$

Fund. Theorem
All $\mathbb{Z}$ are a product of primes

PULVERIZER:
x y x%y = x − qy

GCD Properties:
1) Every common divisor divides the GCD
2) GCD(ka, kb) = k GCD(a, b)
3) GCD(a, b) = 1 and GCD(a, c) = 1 → GCD(a, bc) = 1
4) a | bc & GCD(a, b) = 1 → a | c
5) GCD(a, b) = gcd(b, rem(a, b))

SUMS & ASSYMPTOTES
$H_n = \sum_{i}^{n} \frac{1}{i} \doteq \ln(n)$
convert $\prod \to \sum$ with ln
$n! \doteq \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

COUNTING
Pigeonhole Principle
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1!\, k_2!\, k_3!}$
BOOKKEEPER RULE

evenness $= \frac{1}{1-x^2}$
$\leq 4 = \frac{1-x^5}{1-x}$

INCLUSION-EXCLUSION:
$|S_1 \cup S_2 \cup S_3 \cup \ \ \cup S_n|$
Sum of sets
− sum of 2-way ×
+ sum of 3-way ×
− sum of ...

GENERATING FUNCTIONS
$\langle 1, 1, 1, \ldots \rangle = \frac{1}{1-x}$ R-Shift by x^k
$\langle 1, -1, 1, -1, \ldots \rangle = \frac{1}{1+x}$ AB = convolution a∗b
$\langle 1, a, a^2, a^3, \ldots \rangle = \frac{1}{1-ax}$
$\langle 1, 0, 1, 0, \ldots \rangle = \frac{1}{1-x^2}$
$\left\langle \binom{k}{0}, \binom{k}{1}, \ldots \binom{k}{k}, 0, 0, \ldots \right\rangle = (1+x)^k$

PROBABILITY

4 Step Method:
1) Find Sample Space
2) Define Events → Create tree
3) Determine Outcome Probabilities

Pr{B}

Birthday Principle:
$e^{-\frac{\binom{n}{2}}{d}}$ is max prob of no birthdays (approx)