Scott Young, Sept 18, 2012

PROBLEM ONE 10/15

a) Combine the two image irradiance equations to eliminate $\rho(x,y)$

$$E_1(x,y) = \rho(x,y)R_1(x,y)$$
$$E_2(x,y) = \rho(x,y)R_2(x,y)$$
$$\rho(x,y) = \frac{E_2(x,y)}{R_2(x,y)}$$

✓

$$\boxed{E'(x,y) = \frac{E_2(x,y)}{R_2(x,y)} \cdot R'(x,y)}$$

b) Suppose the underlying surface is Lambertian. Show that the isophotes in gradient space are straight lines. When will they be parallel?

For Lambertian, reflectance map is the normalized dot product:

$$R(p,q) = \frac{r \cdot s}{\|r\|\|s\|} = \frac{1 + p_s p + q_s q}{\sqrt{1+p^2+q^2}\sqrt{1+p_s^2+q_s^2}}$$

$$E'(x,y) = \frac{\dfrac{r \cdot s_1}{\cancel{\|r\|}\|s_1\|}}{\dfrac{r \cdot s_2}{\cancel{\|r\|}\|s_2\|}} \cdot E_2(x,y) \qquad \frac{E_1(x,y)}{E_2(x,y)} = \frac{1 + p_{s1}p + q_{s1}q}{1 + p_{s2}p + q_{s2}q} \cdot \frac{\|s_2\|}{\|s_1\|}$$

isophotes occur at contours of constant brightness

Let c = "scaled" brightness ratio $= \dfrac{\|s_1\| E_1}{\|s_2\| E_2}$

$$c = \frac{\|s_1\|}{\|s_2\|}\frac{E_1}{E_2} = \frac{1 + p_{s1}p + q_{s1}q}{1 + p_{s2}p + q_{s2}q}$$

✓

$$c(1 + p_{s2}p + q_{s2}q) = 1 + p_{s1}p + q_{s1}q$$

$$\begin{matrix} c & + & c p_{s2} p & + & c q_{s2} q \\ -1 & & - p_{s1} p & & - q_{s1} q \end{matrix} = 0$$

$$c - 1 + (c p_{s2} - p_{s1})p + (c q_{s2} - q_{s1})q = 0$$

$$q = -\frac{c p_{s2} - p_{s1}}{c q_{s2} - q_{s1}} p - \frac{c-1}{c q_{s2} - q_{s1}}$$

this forms a straight line in pq or gradient space.

These lines will be parallel when:

$$m_1 = m_2$$

$$\frac{c_0 p_{s2} - p_{s1}}{c_0 q_{s2} - q_{s1}} = \frac{c_1 p_{s2} - p_{s1}}{c_1 q_{s2} - q_{s1}} \quad \text{for } c_0 \neq c_1$$

$$(c_0 p_{s2} - p_{s1})(c_1 q_{s2} - q_{s1}) = (c_0 q_{s2} - q_{s1})(c_1 p_{s2} - p_{s1})$$

$$\cancel{c_1 c_0 p_{s2} q_{s2}} - c_0 p_{s2} q_{s1} - c_1 p_{s1} q_{s2} + \cancel{p_{s1} q_{s1}} = \cancel{c_1 c_0 p_{s2} q_{s2}} - c_0 q_{s2} p_{s1} - c_1 q_{s1} p_{s2} + \cancel{p_{s1} q_{s1}}$$

$$c_0 p_{s2} q_{s1} + c_1 p_{s1} q_{s2} = c_0 q_{s2} p_{s1} + c_1 q_{s1} p_{s2}$$

$$p_{s2} q_{s1} \cancel{(c_0 - c_1)} = q_{s2} p_{s1} \cancel{(c_0 - c_1)}$$

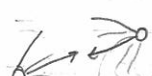
$$\boxed{\frac{p_{s2}}{q_{s2}} = \frac{p_{s1}}{q_{s1}}}$$

i.e. the two sources point in the same direction.

---

c) Show that the isophotes all go through a common point in gradient space, in the case that they are not parallel. Where in gradient space would you expect the highest accuracy? How should you set up the light sources?

Claim: There exists a point (p_0, q_0) such that all isophotes intersect

We should set up the light sources so that they face each other  this results in the greatest accuracy in gradient space, but must be weighed against self-shadowing because the mathematics ignore this.

PROBLEM TWO 12/25

a) Show that the parameters a-f can be estimated with $M, M_x, M_y, M_{xx}, M_{xy}$ and M_{yy} evaluated at the origin

2/4

$$M(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$$

$$\frac{1}{2}M_{xx} \quad \cancel{2}M_{xy} \quad \frac{1}{2}M_{yy} \quad M_x \quad M_y \quad M$$

$M_x(x,y) = 2ax + by + c$
$M_y(x,y) = bx + 2cy + e$
$M_{xx}(x,y) = 2a$
$M_{xy}(x,y) = b$
$M_{yy} = 2c$

b) Match stencils with parameters a-f

4/4

$k_1 \begin{bmatrix} +1 & +1 & +1 \\ -2 & -2 & -2 \\ +1 & +1 & +1 \end{bmatrix} \longrightarrow M_{yy}$

$k_2 \begin{bmatrix} -1 & & +1 \\ & & \\ +1 & & -1 \end{bmatrix} \longrightarrow M_{xy}$

$k_3 \begin{bmatrix} +1 & +1 & +1 \\ & & \\ -1 & -1 & -1 \end{bmatrix} \longrightarrow M_y$

$k_4 \begin{bmatrix} -1 & +2 & -1 \\ +2 & +5 & +2 \\ -1 & +2 & -1 \end{bmatrix} \longrightarrow M$

$k_5 \begin{bmatrix} -1 & & +1 \\ -1 & & +1 \\ -1 & & +1 \end{bmatrix} \longrightarrow M_x$

$k_6 \begin{bmatrix} +1 & -2 & +1 \\ +1 & -2 & +1 \\ +1 & -2 & +1 \end{bmatrix} \longrightarrow M_{xx}$

c) Find the values of k_{1-6}

$k_1 = 1/6\epsilon^2$
$k_2 = \frac{1}{4\epsilon^2}$
$k_3 = 1/6\epsilon$

$k_4 = 1/9$
$k_5 = 1/6\epsilon$
$k_6 = 1/6\epsilon^2$

1/3

d) Show that the distance from the origin to the maximum in gradient is:

$$\rho = -\frac{M_x E_x + M_y E_y}{M_{xx}E_x^2 + 2M_{xy}E_xE_y + M_{yy}E_y^2}\sqrt{E_x^2 + E_y^2}$$

$\nabla \cdot (M_x, M_y)$ = first directional derivative in direction of local gradient $= M_xE_x + M_yE_y$

0/5

$$\begin{pmatrix} E_x & E_y \end{pmatrix}\begin{pmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = M_{xx}E_x^2 + 2M_{xy}E_xE_y + M_{yy}E_y^2$$

second directional derivative of the local gradient magnitude in the direction of the gradient

$$-\frac{\nabla \cdot (M_xM_y)}{\nabla^T H(M) \cdot \nabla} \cdot \frac{E_x + E_y}{\sqrt{M_x + M_y}} \qquad = \frac{1}{\sqrt{E_x + E_y}}$$

Sign b/c

should be negative and num. should be positive.

e) ...

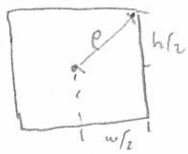
5/5

This is because the brightness gradient must be increasing at a decreasing rate to have a point of maximal brightness

f) How far away can ρ reasonably become?

At most ρ could equal $\sqrt{(w/2)^2+(h/2)^2}$ with w and h equal to the image width and height respectively, corresponding to a local maxima at the corner:

0/4



PROBLEM THREE 19/20

3a) a) Show $s_x = 0$ and $s_y = 0$

$$E(x,y) = R(p,q) = p^2 + q^2$$

4/4

$E_x = 0$, $E_y = 0$ occurs at an extrema of the function, however, the only extrema of $R(p,q)$ occurs at $p=0, q=0$ for which $R_p = R_q = 0 = s_x = s_y$.

b) Show $E_{xx} - 2E_{xy} + E_{yy} = k((a-b)^2 + (b-c)^2)$

$$E = R = \frac{\partial z}{\partial x}^2 + \frac{\partial z}{\partial y}^2$$

by definition

$$p = \frac{\partial z}{\partial x} = 2ax + 2by$$

3/3

$$q = \frac{\partial z}{\partial x} = 2bx + 2cy$$

$$E = (2ax+2by)^2 + (2bx+2cy)^2$$
$$= 4\left[(a^2x^2 + 2abxy + b^2y^2) + (b^2x^2 + 2bxcy + c^2y^2)\right]$$

$$E_{xx} = 8a^2 + 8b^2$$
$$E_{xy} = 8ab + 8bc$$
$$E_{yy} = 8b^2 + 8c^2$$

$$(a-b)^2 + (b-c)^2 = a^2 - 2ab + b^2 + b^2 - 2ac + c^2$$

$$8(a^2+b^2 - ab - bc + b^2 + c^2) = k(a^2 - 2ab + 2b^2 - 2ac +$$

$$\boxed{k=8}$$

c) If $E_{xx} = E_{xy} = E_{yy} = 8$, find a, b and c

$$8 = 8a^2 + 8b^2 \qquad 1 = a^2 + b^2$$
$$8 = 8ab + 8bc \qquad 1 = ab + bc$$
$$8 = 8b^2 + 8c^2 \qquad 1 = b + c$$

$\boxed{a = b = c = \pm\frac{1}{\sqrt{2}}}$

2/3

$$\frac{1}{\sqrt{2}}(8-8) = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)8 \quad ✓$$
$$0 = 0$$

3b) Show that $E(x,y) - R(p,q)$ is constant along the characteristic strip

the characteristic strip consists of 5 differential equations:

$$\dot{x} = R_p \qquad \dot{y} = R_q \qquad \dot{z} = pR_p + qR_q$$
$$\dot{p} = E_x \qquad \dot{q} = E_y$$

where the $(\cdot)$ signifies $\frac{d}{d\xi}$

As the characteristic strip is parameferized by ξ we can conclude that if:

$$= \frac{d}{d\xi}\left(E(x,y) - R(p,q)\right) = 0 \quad \text{the value is constant}$$

$$= \frac{\partial E}{\partial x}\frac{dx}{d\xi} + \frac{\partial E}{\partial y}\frac{dy}{d\xi} - \left(\cancel{\frac{\partial R}{\partial p}\frac{dp}{d\xi}} + \frac{\partial R}{\partial q}\frac{dq}{d\xi}\right)$$

$$= E_x R_p + E_y R_q - \left(R_p E_x + R_q E_y\right)$$

this does $= 0$, Q.E.D.

10/10

PROBLEM FOUR 20/20

a) Show the following:

i) $E_x = f'(r)\frac{x}{r}$ and $E_y = f'(r)\frac{y}{r}$

$$E_x = \frac{\partial}{\partial x}f(r) = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right) = f'(r)\left(\frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2})x\right)$$

$1/r$

$$= \boxed{f'(r)\frac{x}{r}}$$

4/4

$$E_y = \frac{\partial}{\partial y}f(r) = f'(r)\frac{\partial r}{\partial y} = f'(r)\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right) = \boxed{f'(r)\frac{y}{2}}$$

ii) $E_{xx} = f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r}\frac{y^2}{r^2}$, $E_{xy} = f''(r)\frac{xy}{r^2} - \frac{f'(r)}{r}\frac{xy}{r^2}$, $E_{yy} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r}\frac{x^2}{r^2}$

$$E_{xx} = \frac{\partial}{\partial x}E_x = \frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right)$$

$$= \frac{\partial}{\partial x}f'(r)\frac{x}{r} + f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= f''(r)\frac{\partial r}{\partial x}\frac{x}{r} + f'(r)\left(\frac{r - x\cdot x/r}{r^2}\right)$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{x^2}{r^3}\right)$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{x^2}{r^2}\cdot\frac{1}{r}\right)$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{1}{r} + \frac{y^2}{r^2}\cdot\frac{1}{r}\right)$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\left(\frac{r^2 - r^2 + y^2}{r^2\cdot r}\right)$$

$$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r}\left(\frac{r^2 - r^2 + y^2}{r^2}\right)$$

$$= \boxed{f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r}\left(\frac{y^2}{r^2}\right)}$$

$r^2 = x^2 + y^2$

$y^2 = r^2 - x^2$

$1 = \frac{y^2}{r^2} + \frac{x^2}{r^2}$

$1 - \frac{x^2}{r^2} = \frac{y^2}{r^2}$

$1 - \frac{xy}{r^2} = \frac{xy}{r^2}$

$r^2 - xy = xy$

$r^2 - 2xy = 0$

$r^2 = 2xy$

$$E_{xy} = \frac{\partial}{\partial y}E_x = \frac{\partial}{\partial y}\left(f'(r)\frac{x}{r}\right)$$

$$= f''(r)\frac{\partial r}{\partial y}\cdot\frac{x}{r} + f'(r)\frac{\partial}{\partial y}\left(\frac{x}{r}\right) = f''(r)\frac{xy}{r^2} + \frac{f'(r)}{r}\left(0 - \frac{xy}{r^2}\right)$$

$$= f''(r)\frac{yx}{r} + f'(r)\left(\frac{0 - \frac{xy}{r}}{r^2}\right) \quad \boxed{= f''(r)\frac{xy}{r^2} - \frac{f'(r)}{r}\frac{xy}{r^2}}$$

$$E_{yy} = \frac{\partial}{\partial y} E_y = \frac{\partial}{\partial y}\left(f'(r)\frac{y}{r}\right)$$

$$= f''(r)\frac{y^2}{r^2} + f'(r)\frac{\partial}{\partial y}\left(\frac{y}{r}\right)$$

$$= f''(r)\frac{y^2}{r^2} + f'(r)\left(\frac{r - y^2/r}{r^2}\right)$$

$$= f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r}\left(1 - \frac{y^2}{r^2}\right)$$

$$\boxed{= f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r}\frac{x^2}{r^2}}$$

b) Show the following:

i) $\begin{pmatrix} E_x & E_y \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \left(f'(r)\right)^2$

$$= E_x^2 + E_y^2 = \left(f'(r)\frac{x}{r}\right)^2 + \left(f'(r)\frac{y}{r}\right)^2 \qquad r^2 = x^2 + y^2$$

$$= f'^2(r)\frac{x^2}{r^2} + f'^2(r)\frac{y^2}{r^2}$$

4/4

$$= \left(f'(r)\right)^2\left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right)$$

$$= \left(f'(r)\right)^2(1) = \boxed{\left(f'(r)\right)^2}$$

ii) $\begin{pmatrix} E_y & -E_x \end{pmatrix}\begin{pmatrix} E_{xx} & E_{xy} \\ E_{xy} & E_{yy} \end{pmatrix}\begin{pmatrix} E_y \\ -E_x \end{pmatrix} = \frac{\left(f'(r)\right)^3}{r}$

$$= \begin{pmatrix} E_{xx}E_y - E_{xy}E_x & E_{xy}E_y - E_x E_{yy} \end{pmatrix}\begin{pmatrix} E_y \\ -E_x \end{pmatrix}$$

$$= E_y^2 E_{xx} - E_{xy}E_x E_y - E_x E_y E_{xy} + E_x^2 E_{yy}$$

$$= E_y^2 E_{xx} - 2E_{xy}E_x E_y + E_x^2 E_{yy}$$

$$= \left(f'(r)\right)^2\left(\frac{y^2}{r^2}\right)\left(f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r}\frac{y^2}{r^2}\right) - 2\left(f''(r)\frac{xy}{r^2} - \frac{f'(r)}{r}\frac{xy}{r^2}\right)\left(f'(r)\frac{x}{r}\right)\left(f'(r)\frac{y}{r}\right) + \left(f'(r)\right)^2\left(\frac{x^2}{r^2}\right)\left(f''(r)\frac{y^2}{r^2} + \frac{f'}{r}\right.$$

$$= \left(f'(r)\right)^2\left(\frac{1}{r^4}\right)\left(\cancel{f''(r)x^2y^2} + \frac{f'(r)}{r}y^4 - \cancel{2(f''(r)x^2y^2} - \frac{f'(r)}{r}x^2y^2) + \cancel{f''(r)x^2y^2} + \frac{f'(r)}{r}x^4\right)$$

$$= (f'(r))^2\left(\frac{1}{r^4}\right)\left(\frac{f'(r)}{r}y^4 - 2\frac{f'(r)}{r}x^2y^2 + \frac{f'(r)}{r}x^4\right)$$

$$= (f'(r))^3\left(\frac{1}{r^5}\right)\left(y^4 - 2x^2y^2 + x^4\right)$$

$$= (f'(r))^3\left(\frac{1}{r^5}\right)\left(y^2 - x^2\right)^2$$

$$= (f'(r))^3\left(\frac{1}{r^5}\right)\left(r^2\right)^2 = \boxed{\frac{(f'(r))^3}{r}}$$

---

iii) $$\iint_D \frac{E_{xx}E_y^2 - 2E_{xy}E_xE_y + E_{yy}E_x^2}{E_x^2 + E_y^2}\,dx\,dy = -2\pi$$

$$\iint_D \frac{(f'(x))^3/r}{(f'(r))^2}\,dx\,dy = \iint_D \frac{f'(r)}{r}\,dx\,dy$$

$$= \iint_D \frac{f'(r)}{\cancel{r}}\cancel{r}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{R+\epsilon} f'(r)\,dr\,d\theta$$

$$= \int_0^{2\pi} f(r)\Big|_0^{R+\epsilon}\,d\theta$$

$$= \int_0^{2\pi} -1\,d\theta = -\theta\Big|_0^{2\pi} = \boxed{-2\pi}$$

---

c) How does this result change if we repace $E(x,y)$ with $1-E(x,y)$?

---

i) $E(x,y) = 1 - f(r)$ $\quad E_x = -f'(r)\frac{x}{r}$ $\quad E_y = -f'(r)\frac{x}{r}$

$E_{xx} = -\left(f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r}\frac{y^2}{r^2}\right)$ ...

all partial derivatives are now negative.

i.e. $E_x' = -E_x$

4/4

Because $-E_{xx}E_y^2 + 2E_{xy}E_xE_y - E_{yy}E_x^2 = -\frac{(f'(r))^3}{r}$, the result is now $= \boxed{+2\pi}$

ii) What if there is an interior, dark circle?

Now the result $= 0$ because $\displaystyle\int_0^{2\pi} \underbrace{f(r)\Big|_0^{R+\epsilon}}_{\text{evaluates to}} d\theta$

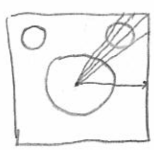
$$\underset{\substack{\| \\ 0}}{f(R+\epsilon)} - \underset{\substack{\| \\ 0}}{f(0)}$$

iii) What if there is more than one bright disc?

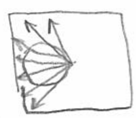
Now $f(r,\theta)$ or $f(x,y)$ not just $f(r)$

we can consider $f'(r,\theta)$ to be modelled as an impulse function with ↑ for each rising edge of f and ↓ for each downward edge.

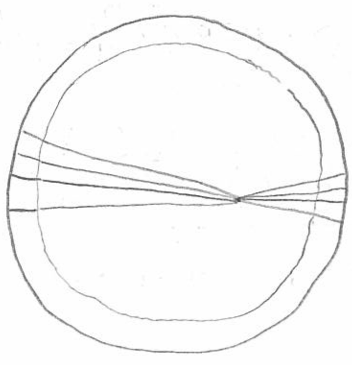
$\iint_D \cdots \, dr\,d\theta$ then computes the angular 'rise and fall'



If all circles are contained fully within the image the result will be 2π for each circle $\therefore$ $\boxed{= -n\,2\pi}$

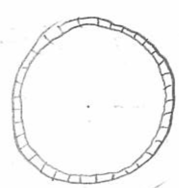


d) What do you think is being computed when applied to a blurry image?



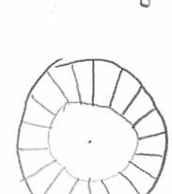
Instead of computing areas this computes radial 'slices through the image's disk edges.

4/4



for $\epsilon \to 0$

$f(r)\Big|_0^{R+\epsilon} = -1$



for $\epsilon >> 0$

$f(r)\Big|_0^{R+\epsilon} = -1$ still

Therefore this measure is total curvature in radians and is independent of blur. ← a.k.a. Euler Number

e) Show that the integrand is the second derivative of brightness along a particular direction. Which direction?

The integrand can be simplified to:

$$\frac{(E_y \;\; -E_x)\begin{pmatrix} E_{xx} & E_{xy} \\ E_{xy} & E_{yy} \end{pmatrix}\begin{pmatrix} E_y \\ -E_x \end{pmatrix}}{\|(E_y \;\; -E_x)\|^2}$$

4/4

This is equivalent to:

$$\frac{(E_y \; -E_x)\, H(E) \begin{pmatrix} E_y \\ -E_x \end{pmatrix}}{E_y^2 + E_x^2}$$

where $H(E)$ is the Hessian matrix of E and $\boxed{(E_y \;\; -E_x)}$ is the direction of the 2nd derivative

This is perpendicular to the gradient.

PROBLEM FIVE 3/10

5 a) a)

If the output of the first accumulator is fed into the second we will get

3/3

$$out = \sum_{i=0}^{N} i f(i) \quad \text{where } f(i) = \begin{cases} 1 & \text{if pixel shaded} \\ 0 & \text{else} \end{cases}$$

b) We can calculate the first moment by modifying the second counter to have two parts. The first part counts the remainder, rolling over every time it exceeds n. The second part is incremented on each rollover, this stores the first moment.

0/3

c) Further moments can be calculated by feeding this input into an identical rollover counter.

0/4

$\boxed{5B}$ The error will go down quadratically. Consider:

As defined the error is some standard deviation equalling a multiple of the size of the pixel.

$\sigma = k\epsilon$ where k is the constant multiple and ϵ is pixel size

Now as the circle radius increases we average the row centroid estimates, which by the central limit theorem tells us $\sigma' = \frac{\sigma}{r}$ as r = # of samplings. Since pixel size ϵ is itself inversely proportional to r this results in a quadratic reduction in error if error is taken to be relative to disc size. If it is an absolute term (i.e. expected # of pixels) then the error will reduce linearly.

by sqrt factor

GRADING:

PROBLEM ONE:	10/15
TWO:	12/25
THREE:	19/20
FOUR:	20/20
FIVE:	3/20
	64/100 = 64%

6.801 - Machine Vision

QUIZ TWO

Scott Young Sept 21, 20

GRADE

ONE: 17/20
TWO: 4/15
THREE: 24/30
FOUR: 11/15
FIVE: 4/20

65.1°

PROBLEM ONE 17/20

a) Find best-fit translation

$$r_r = sR(r_e) + r_o$$

3/3

$$\frac{d}{dr_o}\left(\sum_{i=0}^{N} \| r_{ri} - sR(r_{ei}) - r_o \|^2\right) = 0$$

$$\sum_{i=0}^{N-1} \cancel{2}\left(r_{ri} - sR(r_{ei}) - r_o\right)\cancel{(-1)} = 0$$

$$\sum_{i=0}^{N-1} r_{ri} - \sum_{i=0}^{N-1} sR(r_{ei}) - r_o \cdot N = 0$$

$$N \cdot \bar{r}_r - NsR(\bar{r}_e) - Nr_o = 0$$

$$+\bar{r}_r - sR(\bar{r}_e) - r_o = 0 \qquad \boxed{r_o = \bar{r}_r - sR(\bar{r}_e)}$$

b) By shifting the origin of each measurement to its centroid, show that the translation factor drops out

$r'_{ri} = r_{ri} - \bar{r}_r$

$r_{ri} = \bar{r}_r - r'_{ri}$

3/3

$$\sum_{i=0}^{N-1} \| -r'_{ri} + \bar{r}_r - sR(-r'_{ei} + \bar{r}_e) - (\bar{r}_r - sR(\bar{r}_e)) \|^2$$

$$= \sum_{i=0}^{N-1} \| r'_{ri} - sR(r'_{ei}) + \cancel{sR(\bar{r}_e)} - \cancel{\bar{r}_r} + \cancel{\bar{r}_r} - \cancel{sR(\bar{r}_e)} \|^2$$

$$\boxed{= \sum_{i=0}^{N-1} \| r'_{ri} - sR(\bar{r}_{ei}) \|^2}$$

c) i) Show that this equals $S_r - 2sD_{re} + s^2 S_e$

$$= \sum_{i=0}^{N-1}\left(r'_{ri} \cdot r'_{ri} - 2s\, R(r'_{ei}) \cdot r'_{ri} + s^2 R(r'_{ei}) R(r'_{ei})\right)$$

3/3

$$S_r \quad - 2s D_{er} \quad + s^2 S_e$$

this is true because R is an orthonormal matrix, leaving the length of $\|R(r_{ei})\| = \|r_{ei}\|$.

$$\boxed{= S_r - 2s D_{er} + s^2 s^2 S_e}$$

c) ii) Conclude best fit scale $= D_{re}/S_e$

$$\frac{d}{ds}\left(S_r - 2s\,D_{re} + s^2 S_e\right)$$

$$= -2D_{re} + 2s\,S_e$$

$$2s\,S_e = 2D_{re}$$

$$\boxed{s = \frac{D_{re}}{S_e}}$$

d) Show $s' = D_{er}/S_r$

$$r_e - s'R'(r_r) + r_0'$$

$$\hookrightarrow \sum_{i=0}^{N-1} \left\| r_{ei}' - s'R'(r_{ri}') \right\|^2$$

$$S_e - 2s'D_{er} + s^2 S_r$$

this follows the exact calculations but with $r_e \Leftrightarrow r_r$ and $D_{re} \Leftrightarrow D_{er}$

$$\boxed{s' = \frac{D_{er}}{S_r}}$$

3/3

e) Show this assymmetry is removed if we instead minimize

$$\sum_{i=0}^{N-1} \left\| \frac{1}{\sqrt{s}} r_{ri}' - \sqrt{s}\,R(r_{ei}') \right\|^2$$

$$= \sum_{i=0}^{N-1}\left(\frac{1}{s}(r_{ri}' \cdot r_{ri}') - 2R(r_{ei}')\cdot r_{ri}' + sR(r_{ei})R(r_{ei})\right)$$

$$= \frac{d}{ds}\left(\frac{1}{s}S_r - 2D_{re} + s\,S_e\right) = -\frac{1}{s^2}S_r + S_e$$

$$=$$

$$S_r = S_e s^2$$

$$\boxed{s = \sqrt{\frac{S_r}{S_e}}}$$

4/4

we can repeat this calculation to show $s' = \frac{1}{s} = \sqrt{\frac{S_e}{S_r}}$

$\therefore$ the assymmetry has been removed.

f) Show that the best fit scale factor doesn't require rotation or translation information

It does not require rotation information because R, and its dependent variable D_{re} are not present in the formula. Translation information is ~~also~~ unnecessary, provided that relative measurements are taken.

g) Does best fit rotation depend on the method used for determining best fit scale?

Method ONE:

$$S_r - 2sD_{re} + s^2 S_e$$

$$\frac{d}{dR}\left(S_r - 2sD_{re} + s^2 S_e\right)$$

$$= -2s\frac{dD_{re}}{dR}$$

Method TWO:

$$\frac{S_r}{s} - 2D_{re} + sS_e$$

$$\frac{d}{dR}\left(\frac{S_r}{s} - 2D_{re} + sS_e\right)$$

$$= -2\frac{dD_{re}}{dR}$$

Yes. Using the first method, best-fit rotation relies on best-fit scale. Using the second, the two are decoupled.

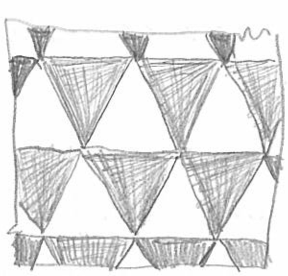
X 0/3

PROBLEM TWO 4/15

a) Explain image processing methods which could be used to recover the vertices

2/4

The image is:

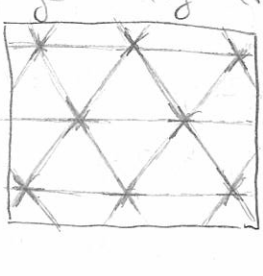


We could first run the image through a filter designed to detect the edges. The Laplacian is useful for this because it is rotationally symmetric and preserves the sign of the brightness change.

We can use the stencil:

$$\frac{1}{6\epsilon^2}\begin{array}{|c|c|c|}\hline 1 & 4 & 1 \\ \hline 4 & -20 & 4 \\ \hline 1 & 4 & 1 \\ \hline\end{array}$$ to approximate the Laplacian locally.

Edges only: (post-filter)



From here we notice that corners will have a larger amount of value per area than edges:

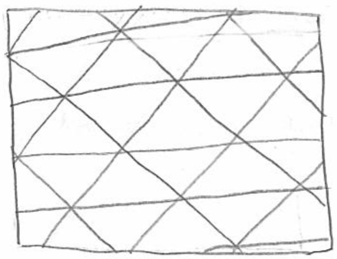
use LS instead

Therefore we can use a set of larger stencils to sum the neighboring area and use non-maximum suppression [illegible]

b) Explain the advantage of this pattern over one with three parallel black lines as edges

---

0/3



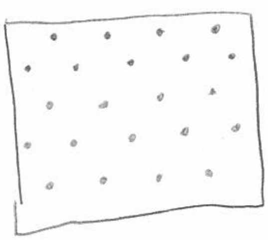
With this pattern we will have a harder time observing brightness distortions caused by the lens ✗

focus/aliasing

---

c) ... a row of black dots?

0/4



Again, brightness distortions will be harder to recognize. ✗ fade, far/away

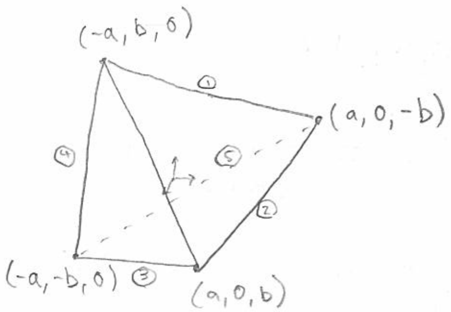
---

d) ... rows of LED lights?

2/4 Brightness distortions again, but also the light will distort the brightness distortions further as the black background is no longer uniform.

PROBLEM THREE 24/30

a) What should the ratio of a:b be for this tetrahedron to be regular?



5/6

All edges must be the same length

① $(-a, b, 0) - (a, 0, -b) \Rightarrow \|(-2a, b, -b)\|^2 = 4a^2 + 2b^2$

② $(a, 0, b) - (a, 0, -b) \Rightarrow \|(0, 0, 2b)\|^2 = 4b^2$

③ $(-a, -b, 0) - (a, 0, b) \Rightarrow \|(-2a, -b, -b)\|^2 = 4a^2 + 2b^2$

④ $(-a, b, 0) - (-a, -b, 0) \Rightarrow \|(0, 2b, 0)\|^2 = 4b^2$

⑤ $(-a, -b, 0) - (a, 0, -b) \Rightarrow \|(-2a, -b, b)\|^2 = 4a^2 + 2b^2$

$4b^2 = 4a^2 + 2b^2$

$2b^2 = 4a^2$

$b^2 = \frac{1}{2}a^2$

$\boxed{b = \frac{1}{\sqrt{2}} a}$ $= \sqrt{2}$ not $\frac{1}{\sqrt{2}}$ $\boxed{-1}$

a) CONT. Find the set of unit quaternions which bring the tetrahedron into self alignment.

centered at origin

we use this vector = -pt as the axis of rotation and we can make 3 rotations per 2π to bring into self alignment

$2\pi/3$

in regular tetrahedron the normalized negative 4th point is the normal for the opposite face

$$\{\mathring{q}_j\} = \left(\sin\frac{\theta_i}{2},\ \cos\frac{\theta_i}{2}\,\hat{p}_k\right)$$ for j = all combination of i and k

for $\theta_i = \frac{2\pi}{3}\cdot i,\ i \in \mathbb{N}$

and $$\hat{p}_k = \frac{\{k\text{th point in tetrahedron} \mid k = 1, 2, 3 \text{ or } 4\}}{\text{length of } k\text{th point}}$$

---

b) Show that for some different axis aliganment, the set of quaternions from (a) can be represented as $\mathring{p}\mathring{q}\mathring{p}^*$, for some unit quaternion $\mathring{p}$

A different axis alignment corresponds to a basis change in the coordinates from one orthonormal basis to another. Basis changes can be made with unit quaternions in the form:

$$\mathring{p}\mathring{r}\mathring{p}^* = \mathring{r}'$$

6/6

rotating the points then amounts to:

$$\mathring{p}\mathring{q}\mathring{p}^*\mathring{r}(\mathring{p}\mathring{q}\mathring{p}^*)^* = \underbrace{\mathring{p}\mathring{q}\mathring{p}^*}_{\text{basis change to the basis of } p}\, r\, \mathring{p}^*\mathring{q}^*\mathring{p}$$

rotation of r by q in new basis

---

c) What is the smallest angle of rotation of the regular polyhedra

---

5 reg. polyhedra : tetrahedron $\longrightarrow 2\pi/3$, smallest rotation

cube $\longrightarrow \pi/2$

octahedron $\longrightarrow \pi/2$

dodecahedron $\longrightarrow 2\pi/5$

isododecahedron $\longrightarrow 2\pi/5$

4/6

d) Show the following: 6/6

i) $(\mathring{a}\mathring{q})\cdot\mathring{b} = \mathring{a}\cdot(\mathring{b}\mathring{q}^*)$

$$\text{LHS} = (aq - \bar{a}\cdot\bar{q},\ a\bar{q} + q\bar{a} + \bar{a}\times\bar{q})\cdot\mathring{b}$$

$$= \underset{1}{\cancel{baq}} - \underset{3}{\cancel{b\bar{a}\cdot\bar{q}}} + \underset{2}{\cancel{a\bar{b}\cdot\bar{q}}} + \underset{4}{\cancel{q\bar{b}\cdot\bar{a}}} - \bar{b}\cdot\bar{a}\times\bar{q}$$

$$\text{RHS} = \mathring{a}\cdot(bq + \bar{b}\cdot\bar{q},\ -\bar{b}\times\bar{q} - b\bar{q} + q\bar{b})$$

$$= (\underset{1}{\cancel{abq}} + \underset{2}{\cancel{a\bar{b}\cdot\bar{q}}} + \bar{a}\cdot\bar{b}\times\bar{q} - \underset{3}{\cancel{b\bar{a}\cdot\bar{q}}} + \underset{4}{\cancel{q\bar{a}\cdot\bar{b}}})$$

$$= +[b\ a\ q] + \dots \qquad = -[a\ b\ q] + \dots$$

these are equivalent by the identity that odd permutations of a triple product sum to zero.

ii) $(\mathring{a}\mathring{b})\cdot(\mathring{a}\mathring{b}) = (\mathring{a}\cdot\mathring{a})(\mathring{b}\cdot\mathring{b})$

$$\text{LHS} = (ab - \bar{a}\cdot\bar{b})^2 + (a\bar{b} + b\bar{a} - \bar{a}\times\bar{b})^2 = (a^2 + \bar{a}\cdot\bar{a})(b^2 + \bar{b}\cdot\bar{b})$$

$$\text{LHS} = (a^2b^2 - \cancel{2ab\bar{a}\cdot\bar{b}} + (\bar{a}\cdot\bar{b})^2) + (a^2\bar{b}\cdot\bar{b} + \cancel{ab\bar{a}\cdot\bar{b}} + \cancel{ab\bar{a}\cdot\bar{b}} + b^2\bar{a}\cdot\bar{a} + (\bar{a}\times\bar{b})^2)$$

$$(\bar{a}\cdot\bar{b})^2 + (\bar{a}\times\bar{b})^2 = \|a\|^2\|b\|^2$$

$$\|a\|^2\|b\|^2\cos^2\theta + \|a\|^2\|b\|^2\sin^2\theta =$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$= a^2b^2 + a^2\|b\|^2 + b^2\|a\|^2 + \|a\|^2\|b\|^2$$

$$\updownarrow \qquad \updownarrow$$

$$\text{RHS} = a^2b^2 + a^2\|b\|^2 + \|a\|^2\|b\|^2 + b^2\|a\|^2$$

iii) $\mathring{r}\mathring{s} = (-r\cdot s,\ r\times s)$

$$(0,\bar{r})(0,\bar{s}) =$$

$$(0\cdot 0 - \bar{r}\cdot\bar{s},\ 0\bar{r} + 0\bar{s} + \bar{r}\times\bar{s}) = (-\bar{r}\cdot\bar{s},\ \bar{r}\times\bar{s})$$

iv) $(\mathring{r}\mathring{s})\cdot\mathring{t} = [\mathring{r}\mathring{s}\mathring{t}]$

$$(-t(\bar{r}\cdot\bar{s}),\ \bar{r}\times\bar{s}\cdot\bar{t}) = (-t\bar{r}\cdot\bar{s},\ \bar{r}\cdot\bar{s}\times\bar{t})$$

by equality of even permutations

e) Show that $r' = (q^2 - \bar{q}\cdot\bar{q})r + 2q(\bar{q}\times\bar{r}) + 2(\bar{q}\cdot\bar{r})\bar{q}$ can be written as $r' = (q^2 + \bar{q}\cdot\bar{q})r + 2q(\bar{q}\times\bar{r}) + 2\bar{q}\times(\bar{q}\times\bar{r})$ and that the latter requires fewer arithmetic operations to evaluate

---

What is needed is to show

$$-\bar{q}\cdot\bar{q}r + 2(\bar{q}\cdot\bar{r})q = \bar{q}\cdot\bar{q}r + 2\bar{q}\times(\bar{q}\times\bar{r})$$

or that:

3/6

$$(\bar{q}\cdot\bar{r})\bar{q} = \bar{q}\cdot\bar{q}\bar{r} + \bar{q}\times(\bar{q}\times\bar{r})$$

$$\|q\|^2\|r\|\cos\theta_{qr}\,\hat{q} = \|q\|^2\|r\|\hat{r} - (\|q\|\cdot\|q\|\|r\|\sin\theta_{qr})\hat{r}$$

$$\cos\theta_{qr}\hat{q} = \hat{r} - \sin\theta_{qr}\hat{r}$$

true by trig identity:

The latter requires fewer arithmetical operations because:

$(q^2 + \bar{q}\cdot\bar{q}) \equiv 1$, thus eliminating 7 operations (1 for squari[ng], 3 for dot-pro[duct], 3 for multipli[cation] of a vector by scalar

while $\bar{q}\times(\bar{q}\times\bar{r}) \equiv \bar{q}(\bar{q}\cdot\bar{r}) - \bar{r}(\bar{q}\cdot\bar{q})$

which has 13 operations, compared with $2(\bar{q}\cdot\bar{r})\bar{q}$ which has 7

$13 - 7 = 6$

$6 - 7 = -1$, one fewer operatio[n]

PROBLEM FOUR

$\boxed{4a)}$ 3/7

$$x_{P_i} = \frac{x_c f}{z_c} + x_0, \quad y_{P_i} = \frac{y_c f}{z_c} + y_0$$

$$\sum_{i=1}^{N} \left(x_{Ii} - \frac{x_c}{z_c}f - x_0\right)^2 + \left(y_{Ii} - \frac{y_c}{z_c}f - y_0\right)^2$$

$$x_c:\quad 0 = 2\left(x_{I1} - \frac{f}{z_c}x_c - x_0\right)\left(-\frac{f}{z_c}\right)$$

$$x_c = \frac{(x_0 - x_{I1})z_c}{f}$$

$$y_c:\quad y_c = \frac{(y_0 - y_{I1})z_c}{f}$$

$$z_c:\quad 0 = \left(x_{I1} - \frac{x_c}{z_c}f - x_0\right)\left(\frac{x_c}{z_c^2}f\right) + \left(y_{I1} - \frac{y_c}{z_c}f - y_0\right)\left(\frac{y_c}{z_c^2}f\right)$$

$$= \frac{x_{I1}x_c}{z_c^2}f - \frac{x_c^2}{z_c^3}f^2 - \frac{x_0 x_c}{z_c^2}f + \frac{y_{I1}y_c}{z_0^2}f - \frac{y_c^2}{z_c^3}f^2 - \frac{y_0 y_c}{z_c^2}f$$

4a) CONT.

From this we can only estimate $\frac{x_c}{z_c}$, $\frac{y_c}{z_c}$. With more points, i.e. $N > 1$, we can estimate distance too, provided $x_0 \neq x_1$ or $y_0 \neq y_1$ ~~for~~ the two cameras. In general, 5 points are needed before the equations are fully constrained. doesn't change

4b)

8/8

There is no true minimum number of rays because we can always scale the baseline along with the world coordinates and obtain the same solution.

PROBLEM FIVE

Show that the motion field is the same when:

4/20

$t = a \quad n = b \quad \omega = c \quad$ and $\quad t = b \quad n = a \quad \omega = c + k\, a \times b$

Find k.

Diagram illustrating the ambiguity:

$$\dot{r} = \frac{\dot{R}}{\dot{R}\cdot\hat{z}} = \frac{-t - \omega \times R}{(-t - \omega \times R)\cdot\hat{z}}$$

$$(-a - c \times R_0)\cdot b = 0$$

3 equations $\quad (-b - (c + k\,a\times b)\times R)\cdot a = 0$

$$\frac{-a - c\times R_0}{(-a - c\times R_0)\cdot\hat{z}} = \frac{-b - (c + k\,a\times b)\times R_1}{(-b - (c + k\,a\times b)\times R_1)\cdot\hat{z}}$$